MATH 255: Assignment 6 Solutions

1. (a) For the result to be true we must have $a < b$. Since $f \in \mathcal{R}(a,b)$, the set of discontinuities of $f$ is a set of measure zero. Since $[a,b]$ is not of measure zero, there is a point $c \in [a,b]$ such that $f$ is continuous at $c$. Since $f(c) > 0$ and $f$ is continuous at $c$, there is an interval $[c,d] \subseteq [a,b]$ with $c < d$ such that $f(x) > m$ on $[c,d]$. But then $\int_a^b f(x)\,dx \geq \int_c^d f(x)\,dx \geq m(d-c) > 0$.

   (b) Any positive function which is zero except for a finite set of points has integral zero.

2. (a) Let $\epsilon > 0$ be given. Since $f_n \to 0$ uniformly on $S$, there exists $N$ such that $|f_n(x)| < \epsilon$ for $n \geq N$ and all $x \in S$. Since $x_n \in S$, we have $|f_n(x_n)| < \epsilon$ for $n \geq N$. This shows that $f_n(x_n) \to 0$. Notice that we have shown that $f_n \to 0$ uniformly on $S$ implies that $f_n(x_n) \to 0$ for an arbitrary sequence $(x_n)$ in $S$. It is left as an exercise for the reader to show that the converse to this statement holds.

   (b) If $f_n(x) = x^n$, then $f_n \to 0$ on $S = (0,1)$ but the convergence is not uniform since $x_n = 1/\sqrt[n]{2} \in S$ implies that $f_n(x_n) = 1/2$ so that $f_n(x_n)$ does not converge to zero. However, if $x_n$ is any sequence which converges to $a \in S$, there is an $r$ with $a < r < 1$ such that $x_n < r$ for $n \geq N$. Then $f_n(x_n) = x_n^n < r^n$ for $n \geq N$ implies $f_n(x_n) \to 0$.

3. We have $\sin x = x - x^3/3! + x^5/5! - \cdots$ so that $(x - \sin x)/x^3 = 1/6 + R(x)$ with

$$R(x) = \sum_{n=3}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = -\frac{x^5}{5!} + \frac{x^7}{7!} - \cdots,$$

   an alternating series with $x^{2n-1}/(2n-1)!$ a decreasing sequence for $x \leq 1$. Hence $|R(x)| < x^5/5!$ for $x \leq 1$ which shows that $R(x) \to 0$ as $x \to 0$.

4. (a) If $x \neq 0$, we have $f_n(x) = (1+x)/(1+x^{2n}) \to 0$ since $x^{2n} \to \infty$. Since $f_n(0) = 1$, we have $f_n \to f$ with $f(0) = 1$ and $f(x) = 0$ if $x \neq 0$.

   (b) Since $f_n$ is continuous for all $n$ but $f$ is not, the convergence cannot be uniform. This can also be seen by taking $x_n = 1/n$ and noting that $f_n(x_n) = 1$ so that $f_n(x_n)$ does not converge to zero.