MATH 255: Assignment 5 Solutions

1. (a) (f_n) converges pointwise to 0 on $\mathbb{R}$ since $\left|\dfrac{x}{x+n}\right| \le \dfrac{|x|}{|n-|x||} < \epsilon$ for $n > |x| + |x|\epsilon$.

   (b) For $0 \le x \le a$, the convergence is uniform since $\dfrac{x}{x+n} \le \dfrac{x}{n} \le \dfrac{a}{n} < \epsilon$ for $n > \epsilon/a$. To show that the convergence is not uniform for $x \ge 0$, let $\epsilon = 1/2$ and let $N > 0$ be given. Then for $x = N = n$, we have $f_n(x) = 1/2$.

2. (a) Since $f_n(0) = 0$, the sequence converges pointwise for $x = 0$. If $x \ne 0$, we have

$$\left|\frac{nx}{1+n^2x^2}\right| < \frac{1}{n|x|} \le \epsilon \text{ if } n \ge 1/\epsilon|x|$$

   which shows that (f_n) converges pointwise to 0 on $\mathbb{R}$.

   (b) For $x \ge a$, the convergence is uniform since $0 \le f_n(x) < 1/na \le \epsilon$ if $n \ge 1/a\epsilon$. To show that the convergence is not uniform for $x \ge 0$, let $\epsilon = 1/2$ and let $N > 0$ be given. Then for $x = 1/N$, $n = N$, we have $f_n(x) = 1/2$.

3. Let $\epsilon > 0$ be given and pick N so that $g_n(x) < \epsilon$ for all $x \in S$ and all $n \ge N$. Then

$$\left|\sum_{k=n}^{m}(-1)^{k+1}g_k(x)\right| \le g_n(x) < \epsilon$$

   for $m \ge n \ge N$ and all $x \in S$. Thus the series satisfies a uniform Cauchy condition on S and hence converges uniformly on S.