MATH 255: Assignment 4 Solutions

1. Let $M > 0$ with $|f(x)| \leq M$ on $[a, b]$. Let $\epsilon > 0$ be given and choose c so that $a < c \leq b$ and $c - a < \epsilon/2M$. Choose a partition Q of $[c, b]$ such that $U(Q, f) - L(Q, f) < \frac{\epsilon}{2}$ and let $P = Q \cup \{a\}$. Then P is a partition of $[a, b]$ and

$$U(P, f) - L(P, f) = (\sup_{[a,c]} f - \inf_{[a,c]} f)(c - a) + U(P, f) - L(P, f) < 2M(c - a) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

2. (a) f' exists and is bounded on $[0, 1] \implies f$ is of bounded variation on $[0, 1]$

   (b) Let P_n be the partition of $[2/(4n + 1)\pi, 2/\pi]$ formed by points $2/(2k - 1)\pi$ with $1 \leq k \leq 2n$. Then

$$V_f(0, 1) \geq A(P_n) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{2n} \frac{1}{2k - 1} \to \infty \text{ as } n \to \infty$$

   which implies that f is not of bounded variation on $[0, 1]$.

3. $f \geq 0$ and $f \neq 0 \Rightarrow f \geq m > 0$ on $[c, d] \subseteq [a, b]$ with $c \neq d \Rightarrow \int_a^b f(x)\, dx \geq \int_c^d f(x)\, dx \geq m(d - c) > 0$.

4. For any partition $P = \{x_0 = a < x_1 < \cdots < x_n = x\}$ of $[a, x]$,

$$\sum_{k=1}^{n} |\Delta g_k| = \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f(t)\, dt \right| \leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f(t)|\, dt = \int_a^x |f(t)|\, dt$$

which shows that $V_g(a, x) \leq \int_a^x |f(t)|\, dt$. By the MVT for integrals

$$\int_{x_{k-1}}^{x_k} |f(t)|\, dt = c_k \Delta x_k \quad \text{with} \quad m_k(f) \leq c_k \leq M_k(f).$$

We can choose P so that $U(P, f) - L(P, f) < \epsilon$. Pick $t_k \in [x_{k-1}, x_x]$. Then

$$|c_k - f(t_k)| \leq M_k(f) - m_k(f) \implies |c_k| \geq f(t_k) - (M_k(f) - m_k(f))$$

so that

$$V_g(a, x) \geq \sum_{k=1}^{n} |\Delta g_k| = \sum_{k=1}^{n} |c_k| \Delta x_k \geq \sum_{k=1}^{n} |f(t_k)| \Delta x_k - \epsilon \geq L(P, |f|) - \epsilon.$$

The same argument shows that $V_g(a, x) \geq L(Q, |f|) - \epsilon$ for any partition Q finer than P from which we get

$$V_g(a, x) \geq \int_a^x |f(t)|\, dt - \epsilon.$$

The result follows since ϵ is arbitrary.