MATH 255: Assignment 4 Solutions

1. Let $M > 0$ with $|f(x)| \le M$ on $[a,b]$. Let $\epsilon > 0$ be given and choose c so that $a < c \le b$ and $c - a < \epsilon/2M$. Choose a partition Q of $[c,b]$ such that $U(Q,f) - L(Q,f) < \frac{\epsilon}{2}$ and let $P = Q \cup \{a\}$. Then P is a partition of $[a,b]$ and

$$U(P,f) - L(P,f) = (\sup_{[a,c]} f - \inf_{[a,c]} f)(c-a) + U(P,f) - L(P,f) < 2M(c-a) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

2. (a) f' exists and is bounded on $[0,1] \implies f$ is of bounded variation on $[0,1]$

   (b) Let P_n be the partition of $[2/(4n+1)\pi, 2/\pi]$ formed by points $2/(2k-1)\pi$ with $1 \le k \le 2n$. Then

$$V_f(0,1) \ge A(P_n) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{2n} \frac{1}{2k-1} \to \infty \text{ as } n \to \infty$$

   which implies that f is not of bounded variation on $[0,1]$.

3. $f \ge 0$ and $f \ne 0 \Rightarrow f \ge m > 0$ on $[c,d] \subseteq [a,b]$ with $c \ne d \Rightarrow \int_a^b f(x)\,dx \ge \int_c^d f(x)\,dx \ge m(d-c) > 0$.

4. For any partition $P = \{x_0 = a < x_1 < \cdots < x_n = x\}$ of $[a,x]$,

$$\sum_{k=1}^{n} |\Delta g_k| = \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f(t)\,dt \right| \le \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f(t)|\,dt = \int_a^x |f(t)|\,dt$$

   which shows that $V_g(a,x) \le \int_a^x |f(t)|\,dt$. By the MVT for integrals

$$\int_{x_{k-1}}^{x_k} |f(t)|\,dt = c_k \Delta x_k \quad \text{with} \quad m_k(f) \le c_k \le M_k(f).$$

   We can choose P so that $U(P,f) - L(P,f) < \epsilon$. Pick $t_k \in [x_{k-1}, x_x]$. Then

$$|c_k - f(t_k)| \le M_k(f) - m_k(f) \implies |c_k| \ge f(t_k) - (M_k(f) - m_k(f))$$

   so that

$$V_g(a,x) \ge \sum_{k=1}^{n} |\Delta g_k| = \sum_{k=1}^{n} |c_k| \Delta x_k \ge \sum_{k=1}^{n} |f(t_k)| \Delta x_k - \epsilon \ge L(P,|f|) - \epsilon.$$

   The same argument shows that $V_g(a,x) \ge L(Q,|f|) - \epsilon$ for any partition Q finer than P from which we get

$$V_g(a,x) \ge \int_a^x |f(t)|\,dt - \epsilon.$$

   The result follows since ϵ is arbitrary.