MATH 255: Assignment 3 Solutions

1. By induction, it suffices to consider the case that f and g differ at a single point c. It suffices to show that $h = f - g \in \mathcal{R}(a,b)$ and $\int_a^b h(x)\,dx = 0$. Since f is bounded, so is g and hence h. Choose $M > 0$ so that $|h(x)| \le M$ on $[a,b]$. Let $\epsilon > 0$ be given and let $P = \{a = x_0 < x_1 < \cdots < x_n\}$ be a partition of $[a,b]$ of norm $\delta < \epsilon/M$. If $c \in [x_{k-1}, x_k]$ and t is any tag for P, we have $S(P,t,h) = h(t_k)\delta x_k$ so that $|S(P,t,h)| \le M\delta < \epsilon$. If follows that $h \in \mathcal{R}(a,b)$ and $\int_a^b h(x)\,dx = 0$.

2. (a) The Riemann-Stieltjes sum for the partition

$$a, a + \frac{b-a}{n}, \frac{2(b-a)}{n}, \ldots, a + \frac{n(b-a)}{n} = b$$

of $[a,b]$ and tags $a + \frac{b-a}{n}, \frac{2(b-a)}{n}, \ldots, a + \frac{(n(b-a)}{n} = b$ is

$$S_n = \sum_{k=1}^{n} f(a + \frac{k(b-a)}{n})\left(\frac{b-a}{n}\right) = \left(\frac{b-a}{n}\right)\sum_{k=1}^{n} f(a + \frac{k(b-a)}{n}).$$

Given ϵ there is a $\delta > 0$ so that any Riemann-Stieltjes sum S for any tagged partition of norm $< \delta$ satisfies $|S - \int_a^b f(x)\,dx| < \epsilon$. Since the norm of the above partition is $(b-a)/n$, we see that $|S_n - \int_a^b f(x)\,dx| < \epsilon$ if $(b-a)/n < \delta$ or, equivalently $n > N = (b-a)/\delta$. This shows $\lim S_n = \int_a^b f(x)\,dx$.

(b)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + k^2/n^2} = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}.$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k^2}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1 + k^2/n^2}} = \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \log(1 + \sqrt{2}).$$

3. (a) Define f on $[0,1]$ by $f(x) = -1$ if x is rational and $f(x) = 1$ if x is irrational. Then f is not Riemann integrable on $[0,1]$ since $U(P,f) - L(P,f) = 2$ for any partition P of $[0,1]$. Since $|f|$ is the constant function 1, it is Riemann integrable.

   (b) The function f defined by $f(0) = 0$ and $f(x) = x^2 \sin(1/x^2)$ is differentiable for $x \neq 0$ with $f'(x) = 2x\sin(1/x^2) - 2\cos(1/x^2)/x$ which is unbounded on $(0,1]$. We also have $f'(0) = \lim_{h\to 0+} f(h)/n = 0$ so that f' exists on $[0,1]$ but f' is not Riemann integrable there since f' is unbounded on $[0,1]$.