MATH 255: Assignment 2 Solutions

1. Since $A(x)$ is a step function on $[1,a]$ with jump a_n at the natural number n, we have

$$\begin{aligned}\sum_{n\le a} a_n f(n) &= a_1 f(1) + \int_1^a f(x)\,dA(x)\\ &= a_1 f(1) + A(a)f(a) - A(1)f(1) - \int_1^a A(x)\,df(x)\\ &= A(a)f(a) - \int_1^n aA(x)f'(x)\,dx.\end{aligned}$$

2. (a) The integral exists since x^2 is continuous and $[x]$ is increasing.

$$\int_0^5 x^2\,d[x] = 1^2+2^2+3^2+4^2+5^2 = 55.$$

(b) Since $|x|$ is continuous, $[|x|]$ decreasing on $[-2,0]$ and $[|x|]=[x]$ increasing on $[0,3]$, we have

$$\begin{aligned}\int_{-2}^3 [|x|]\,d|x| &= \int_{-2}^0 [|x|]\,d|x| + \int_0^3 [x]\,dx = \int_2^0 [x]dx + \int_0^3 [x]\,dx = \int_2^3 [x]dx\\ &= 3^2-2^2-\int_2^3 x\,d[x] = 5-3 = 2\end{aligned}$$

(c) Since $x^2+[x]$ is increasing and $|3-x|$ is continuous, the integral exists and

$$\begin{aligned}\int_0^6 (x^2+[x])\,d|3-x| &= \int_0^3 (x^2+[x])\,d(3-x) + \int_3^6 (x^2+[x])\,d(x-3)\\ &= -\int_0^3 x^2\,dx + \int_0^3 x\,d[x] + \int_3^6 x^2\,dx + 18 - \int_3^6 x\,d[x]\\ &= -9+1+2+3+72-9+18-(4+5+6) = 63.\end{aligned}$$

(d) Since x is continuous and $[x]$ is increasing,

$$\int_0^2 (x-[x])\,dx = \int_0^2 x\,dx - \int_0^2 [x]\,dx = 2-4+\int_0^2 x\,d[x] = -2+1+2 = 1.$$

3. (a) $$\sum_{k=1}^n \frac{1}{k^s} = 1 + \int_1^n \frac{d[x]}{x^s} = 1 + \frac{1}{n^s} - \int_1^n [x]\,dx^{-s} = 1 + \frac{1}{n^s} + s\int_1^n \frac{[x]}{x^{s+1}}\,dx$$

(b) $$\sum_{k=1}^n \frac{1}{k} = 1 + \int_1^n \frac{d[x]}{x} = 1 + \int_1^n \frac{dx}{x} + \int_1^n \frac{d([x]-x)}{x} = 1 + \log(n) - \int_1^n \frac{x-[x]}{x^2}dx$$