MATH 255: Assignment 10 Solutions

1. We have $R_n = n(1 - \frac{a_{n+1}}{a_n}) = \frac{6n^2+5n}{4n^2+10n+6} \geq 5/4$ for $n \geq 10$. (Use the fact that $f(x) = \frac{6x^2+5x}{4x^2+10x+6}$ has $f'(x) > 0$ for $x \geq 0$.) Then, by the remainder estimate for Raabe's test, $r_n \leq 4na_{n+1} \leq 2P$, where $P = (1 - \frac{1}{4}) \cdots (1 - \frac{1}{2n+2})$. Now $\log P = \sum_{k=1}^{n+1} \log(1 - \frac{1}{2k}) < -\frac{1}{2} \sum_{k=1}^{n+1} \frac{1}{k} < -\frac{1}{2} \log(n+2)$, which gives $P < 1/\sqrt{n+2}$.

2. The verification of the metric space axioms is straightforward. If (f_n) is a Cauchy sequence for d, then (f_n) and (f'_n) are Cauchy sequences for the uniform metric so that they both satisfy the uniform Cauchy condition. This implies that there are functions f and g such that $f_n \to f$ and $f'_n \to g$ uniformly. But this means $f' = g$ so that $d(f_n, f) \to 0$ and hence that (f_n) converges to f with respect to the metric d.

3. (a) Let X be a connected subset of a metric space S and let f be a continuous mapping of S to a metric space S'. To show that $f(X)$ is connected we can wlog suppose that $X = S$ and $f(X) = S'$. Suppose that $S' = A \cup B$ with A, B open, disjoint subsets of S'. then $S = f^{-1}(A) \cup f^{-1}(B)$. Since S is connected and $f^{-1}(A)$, $f^{-1}(B)$ are open in S, we must have $f^{-1}(A) = \emptyset$ or $f^{-1}(B) = \emptyset$. But this implies $A = \emptyset$ or $B = \emptyset$ so that $S' = f(S)$ is connected.

   (b) If X is not an interval there is a point $c \in \mathbb{R} - X$ such that there are point $x, y \in X$ with $x < c < y$. But then $X = A \cup B$ with $A = X \cap (-\infty), c)$, $B = X \cap (c, \infty)$ non-empty, open disjoint subsets of X which implies that X is not connected.

   Conversely, if X is an interval which is not connected, we have $X = A \cup B$ with A, B open, non-empty, disjoint subsets of X. Let $a, b \in X$ with $a \in A$, $b \in B$. We may assume wlog that $a < b$. Let $c = \sup\{x \in [a,b] \mid [a,x] \subset A\}$. Then $a < c$ since A is open and $[a,b] \subseteq X$. Since A is closed and c is a limit point of A, we must have $c \in A$. But then A open implies the existence of $d \in A$ with $c < d < b$ and $[c,d] \subset A$ which contradicts the definition of c.

   Since the continuous image of an interval is connected, the image must be an interval which yields the intermediate value theorem.

   (c) Suppose that $\mathbb{R}^n = A \cup B$ with A, B open, nonempty, disjoint subsets of $\mathbb{R}^n$. Let $a \in A$, $b \in B$ and let X be the straight line joining a and b. Then X is connected since it is the continuous image of a line segment in $\mathbb{R}$. (Use the parametric equations of the line to see this.) But then X is the disjoint union of the open non-empty subsets $X \cap A$ and $X \cap B$ which contradicts the fact that X is connected.

4. (a) We have $f'(x) = 2e^{-x^2} \int_0^x e^{-t^2} dt$, $g'(x) = -2x \int_0^1 e^{-x^2(t^2+1)} dt = -2xe^{-x^2} \int_0^1 e^{-(xt)^2} dt = -2e^{-x^2} \int_0^x e^{-t^2} dt$ so that $f'(x) + g'(x) = 0$. Hence $f(x) + g(x) = C$ with $C = f(0) + g(0) = 0 + \int_0^1 \frac{dt}{1+t^2} = \tan^{-1}(1) = \pi/4$.

   (b) Use the fact that $\int_0^1 e^{-x^2(t^2+1)} dt < e^{-x^2}$ to show that $g(x) \to 0$ as $x \to \infty$.