MATH 255: Assignment 1 Solutions

1. Let $a < c < b$ and define α on $[a, b]$ by $\alpha(x) = 0$ for $a \le x < c$ and $\alpha(x) = 1$ for $c \le x \le b$. Let f be any function on $[a, b]$.

   (a) **Theorem.** The function f is Riemann-Stieltjes integrable with respect to $\alpha \Leftrightarrow f$ is left continuous at c, in which case $\int_a^b f\, d\alpha = f(c)$.

   Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be any partition of $[a, b]$ containing c, say $x_k = c$. Then, for any tag t for P, we have $S(P, t, f, \alpha) = f(t_k)$ with $x_{k-1} \le t_k \le c$.
   ($\Leftarrow$) Suppose that f is left continuous at c and let $\epsilon > 0$ be given. Choose $\delta > 0$ so that $0 \le c - x < \delta$ implies that $|f(x) - f(c)| < \epsilon$. Now let Q be any partition of $[a, b]$, containing c, which is of norm $< \delta$. Then $Q \subseteq P$ implies

   $$|S(P, t, f, \alpha) - f(c)| = |f(t_k) - f(c)| < \epsilon$$

   since $|t_k - c| < \delta_1$. This proves that $f \in \mathcal{R}(\alpha, a, b)$ and $\int_a^b f\, d\alpha = f(c)$.
   ($\Rightarrow$) Suppose that $f \in \mathcal{R}(\alpha, a, b)$. Let $\epsilon > 0$ be given. Then there is a partition Q of $[a, b]$ containing c and of norm $< \delta_1$ such that for any partition P finer than Q we have

   $$|S(P, t, f\alpha) - S(P, t', f\alpha)| = |f(t_k) - f(t'_k)| < \epsilon$$

   for any t_k, t'_k in the interval $[x_{k-1}, x_k = c]$. Letting $t_k = c$ and setting $\delta = x_{k-1}$, we obtain that $0 \le c - x < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. **QED**

   (b) **Theorem.** The function f is strictly Riemann-Stieltjes integrable with respect to $\alpha \Leftrightarrow f$ is continuous at c.

   Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. If $x_{k-1} < c \le x_k$, we have $S(P, t, f, \alpha) = f(t_k)$. If t' is a tag for P with $t'_k = c$, we have

   $$S(P, t, f\alpha) - S(P, t', f\alpha) = S(P, t, f, \alpha) - f(c) = f(t_k) - f(c).$$

   ($\Rightarrow$) Let $\epsilon > 0$ be given and choose a partition P with c the midpoint of $[x_{k-1}, x_k]$ and with $|S(P, t, f\alpha) - S(P, t', f\alpha)| < \epsilon$. Then $|f(x) - f(c)| < \epsilon$ if $|x - c| < \delta = ||P||/2$.
   ($\Leftarrow$) Let $\epsilon > 0$ be given and choose δ so that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. Then $||P|| < \delta$ implies that $|S(P, t, f, \alpha) - f(c)| = |f(t_k) - f(c)| < \epsilon$. This implies that $f \in \mathcal{R}^*(\alpha, a, b)$.
   QED

2. We will not give the proof since it is essentially the same as the proof of Linearity Theorem A.

3. (a) **Theorem.** Let α be a function on $[a, b]$. Then the constant function $1 \in \mathcal{R}(\alpha, a, b)$ and $\int_a^b d\alpha = \int_a^b 1\, d\alpha = \alpha(b) - \alpha(a)$.

   Proof. For any partition P of $[a, b]$, we have $S(P, t, 1, \alpha) = \alpha(b) - \alpha(a)$. Thus, for any $\epsilon > 0$, we have $|S(P, t, 1, \alpha) - (\alpha(b) - \alpha(a))| = 0 < \epsilon$. **QED**

   (b) **Theorem.** Let f, α be functions on $[a, b]$. If $\int_a^b f\, d\alpha = 0$ for every monotonic f then α is constant.
   Proof. If $f = 1$, we have $0 = \int_a^b f\, d\alpha = \alpha(b) - \alpha(a)$. If $a < c < b$, let f be the function defined by $f(x) = 0$ for $0 \le x < c$ and $f(x) = 1$ for $c \le x \le b$. Then f is increasing and

   $$0 = \int_a^b f\, d\alpha = \alpha(b) - \int_a^b \alpha\, df = \alpha(b) - \alpha(c),$$

   using Theorem 1(a). **QED**