MATH 255: Lecture 9

The Riemann-Stieltjes Integral: Mean-Value Theorems, Fundamental Theorems

Theorem 24: First Mean-Value Theorem for Riemann-Stieltjes Integrals. Let f, α be functions on $[a,b]$ with f bounded and α increasing. If $f \in \mathcal{R}(\alpha,a,b)$ and m, M are respectively the inf and sup of f on $[a,b]$, there exists $c \in [m,M]$ such that

$$\int_a^b f(x)\,d\alpha(x) = c(\alpha(b)-\alpha(a)).$$

If f is continuous, we have $c = f(\xi)$ for some $\xi \in [a,b]$.

Proof. For any tagged partition (P,t) of $[a,b]$ we have

$$m(\alpha(b)-\alpha(a)) \leq S(P,t,f,\alpha) \leq M(\alpha(b)-\alpha(a))$$

which implies that $m(\alpha(b)-\alpha(a)) \leq \int_a^b f(x)\,d\alpha(x) \leq M(\alpha(b)-\alpha(a))$. If $\alpha(a)=\alpha(b)$, any $c \in [m,M]$ will do; otherwise, we take

$$c = \frac{1}{\alpha(b)-\alpha(a)} \int_a^b f(x)\,d\alpha(x).$$

QED

Theorem 25: Second Mean-Value Theorem for Rieman-Stieltjes integrals. If α is continuous on $[a,b]$ and f is increasing on $[a,b]$, there is a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)\,d\alpha(x) = f(a)\int_a^\xi d\alpha(x) + f(b)\int_\xi^b d\alpha(x).$$

Proof. Integrating by parts, we have

$$\int_a^b f(x)\,d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\,df(x).$$

Applying the First Mean-Value Theorem to the integral $\int_a^b \alpha(x)\,df(x)$, we get

$$\int_a^b f(x)\,d\alpha(x) = f(a)(\alpha(\xi)-\alpha(a)) + f(b)(\alpha(b)-\alpha(\xi)),$$

where $\xi \in [a,b]$. **QED**

Theorem 26. Let α be of bounded variation on $[a,b]$ and let $f \in \mathcal{R}(\alpha,a,b)$ with f bounded on $[a,b]$. If we define,

$$F(x) = \int_a^x f(t)\,d\alpha(t)$$

for $a \leq x \leq b$, we have

(a) F is of bounded variation on $[a,b]$;

(b) Every point of continuity of α is a point of continuity of F;

(c) If α is increasing, then $F'(x)$ exists at every point where $\alpha'(x)$ exists and $f(x)$ is continuous. For such x we have

$$F'(x) = f(x)\alpha'(x).$$

Proof. Without loss of generality, we can assume α is increasing. By the First Mean-Value Theorem, we have

$$F(y) - F(x) = \int_x^y f(x)\, d\alpha(x) = c(\alpha(y) - \alpha(x)),$$

where $m = \inf f \leq c \leq \sup f = M$. This yields (a) and (b). To prove (c), divide by $y - x > 0$ and let $y \to x$. Note that $c = f(\xi) \to f(x)$. **QED**

Corollary: First Fundamental Theorem of Integral Calculus. If f is a bounded function on $[a, b]$ and $f \in \mathcal{R}$, then

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

at each point of continuity x of f on $[a, b]$.

Theorem 27: Second Fundamental Theorem of Integral Calculus. If f, F are functions on $[a, b]$ with $f \in \mathcal{R}(a, b)$ and F differentiable on $[a, b]$ such that $F'(x) = f(x)$ on (a, b), then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof. For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$, we have

$$F(b) - F(a) = \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} F'(t_k)\Delta x_k = \sum_{k=1}^{n} f(t_k)\Delta x_k,$$

where $x_{k-1} < t_k < x_k$, using the Mean-Value Theorem for Derivatives. **QED**

Exercise 3. Show that Theorem 27 remains true if the condition $F'(x) = f(x)$ on (a, b) is replaced by $F'(x) = f(x)$ except possibly for a finite set of points.

Theorem 28: Change of Variable in a Riemann Integral. Let f be a continuous function on $[a, b]$ and let g be a continuously differentiable function on $[c, d]$ such that $g(c) = a$, $g(d) = b$. Then

$$\int_a^b f(x)\, dx = \int_c^d f(g(t))g'(t)\, dt.$$

Proof. The functions

$$H(x) = \int_a^{g(x)} f(t)\, dt \quad \text{and} \quad K(x) = \int_c^x f(g(t))g'(t)\, dt$$

have the same derivative $f(g(x))g'(x)$ and are equal at $x = c$. **QED**

Theorem: Second Mean-Value Theorem for Integrals. Let f, g be functions on $[a, b]$ with f increasing and g continuous. Then

$$\int_a^b f(x)g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx + f(b) \int_{\xi}^b g(x)\, dx$$

for some $\xi \in [a, b]$.

Proof. If $G(x) = \int_a^x g(x)\, dx$ and

$$\int_a^b f(x)g(x)\, dx = \int_a^b f(x)\, dG(x) = f(a)(G(\xi) - G(a)) + f(b)(G(b) - G(\xi))$$

for some $\xi \in [a, b]$ by the First Mean-Value Theorem for Riemann-Stieltjes Integrals. **QED**

Corollary: Bonnet's Theorem. Let f, g be functions on $[a, b]$ with $f \geq 0$ increasing, and g continuous. Then

$$\int_a^b f(x)g(x)\,dx = f(b)\int_\xi^b g(x)\,dx$$

for some $\xi \in [a, b]$.

Proof. This follows immediately from the Theorem by re-defining f at a to be 0. The integrals don't change and f is still increasing on $[a, b]$.

Taylor's Theorem with Integral Form of Remainder. Suppose that $f, f', \ldots, f^{(n)}, f^{(n+1)}$ exist on $[a, b]$ and that $f^{(n+1)} \in \mathcal{R}(a, b)$. Then,

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + R_{n+1},$$

where $R_{n+1} = \dfrac{1}{n!}\displaystyle\int_a^b f^{(n+1)}(t)(b-t)^n\,dt$.

Proof. For $n = 0$ this is just the second form of the Fundamental Theorem. If we integrate R_n by parts, we get

$$R_{n+1} = \frac{1}{n!}\int_a^b (b-t)^n df^{(n)}(t) = -\frac{f^{(n)}(a)}{n!}(b-a)^n + R_n$$

and the result follows by induction. **QED**

The Riemann-Stieltjes integral $\int_a^b f\,d\alpha$ can be extended to the case f and α are complex-valued functions. The definitions are exactly the same as for the case of real-valued functions. If $f = f_1 + if_2$, $\alpha = \alpha_1 + i\alpha_2$ are complex-valued functions on $[a, b]$, we have

$$\int_a^b f\,d\alpha = \left(\int_a^b f_1\,d\alpha_1 - \int_a^b f_2\,d\alpha_2\right) + i\Big(\int_a^b f_2\,d\alpha_1 + \int_a^b f_1\,d\alpha_2),$$

whenever all four integrals on the right exist. Using this we can extend most of the important theorems to the complex case if we define such notions as continuity, differentiability and bounded variation componentwise.

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