MATH 255: Lecture 8

The Riemann-Stieltjes Integral: Functions of Bounded Variation

The results we have obtained for increasing integrators α can be extended to an important class of functions, namely, functions of **bounded variation**.

Definition. A function α on $[a, b]$ is said to be of bounded variation on $[a, b]$ if the sums

$$A(P) = \sum_{k=1}^{n} |\Delta\alpha_k| = \sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})|,$$

where $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ varies over all partitions of $[a, b]$, are bounded. In this case,

$$V_\alpha(a, b) = \sup_P A(P)$$

is called the total variation of α on $[a, b]$. If α is increasing, we have $V_\alpha = \alpha(b) - \alpha(a)$.

Note that $P \subseteq Q \Rightarrow S(P) \leq S(Q)$. Indeed, if $x_k < c < x_{k-1}$, we have

$$|\alpha(x_k) - \alpha(x_{k-1})| \leq |\alpha(x_k) - \alpha(c)| + |\alpha(c) - \alpha(x_{k-1})|$$

so that $A(P) \leq A(P \cup c)$. Applying this to the case $Q = \{a < x < b\}$, we get $|\alpha(x) - \alpha(a)| \leq V_\alpha(a, b)$, which shows that a function of bounded variation is bounded.

If α is of bounded variation on $[a, b]$ and $a \leq c \leq d \leq b$, then α is of bounded variation on $[c, d]$ and $V_\alpha(c, d) \leq V_\alpha(a, b)$ since any partition of $[c, d]$ can be extended to a partition of $[a, b]$.

If α is continuous on$[a, b]$ with a bounded derivative on (a, b) then, by the Mean Value Theorem for Derivatives, α is of bounded variation on $[a, b]$. It follows that a piecewise smooth function on$[a, b]$ is also of bounded variation. More generally, if α satisfies the Lipschitz condition $|\alpha(x) - \alpha(y) \leq M|x - y|$ for all $x, y \in [a, b]$, then α is of bounded variation on $[a, b]$

Example 1. The function α on $[0, 1]$ by $\alpha(0) = 0$, $\alpha(x) = \sin(1/x)$ if $x \neq 0$ is not of bounded variation. Indeed, if P is the set of points $2/(2k + 1)\pi$, where $0 \leq k \leq n$, then $A(P) = 2n$.

Example 2. The function α on $[0, 1]$, defined by $\alpha(0) = 0$, $\alpha(x) = x \sin(1/x)$ if $x \neq 0$ is continuous but not of bounded variation. Indeed, for the set P in Example 1, we have

$$A(P) = \frac{\pi}{2}(1 + \frac{1}{2} + \ldots + \frac{1}{n}),$$

which can be arbitrarily large.

Exercise 1. If α, β are of bounded variation on $[a, b]$ and $c \in \mathbb{R}$, show that $c\alpha$, $\alpha + \beta$, $\alpha\beta$ are of bounded variation on $[a, b]$. If there is an $m > 0$ so that $|\alpha(x)| \geq m$ for all $x \in [a, b]$, show that $1/\alpha$ is of bounded variation on $[a, b]$.

Theorem 20. If $a \leq c \leq b$, then α is of bounded variation on $[a, b]$ $\iff$ α is of bounded variation on $[a, c]$ and $[c, d]$, in which case

$$V_\alpha(a, b) = V_\alpha(a, c) + V_\alpha(c, b).$$

Proof. If Q, R are partitions of $[a, c]$, $[c, b]$ respectively, then $Q \cup R$ is a partition of $[a, b]$ and

$$A(Q \cup R) = A(Q) + A(R).$$

Moreover, any partition of $[a, b]$ which contains c is of this form. The theorem follows easily from this; the details are left to the reader. **QED**

If α is of bounded variation on $[a, b]$ and $a \le x \le b$, we define $V_\alpha(x) = V_\alpha(a, x)$. If $a \le x \le y \le b$, we have

$$V_\alpha(y) = V_\alpha(x) + V_\alpha(x, y),$$

which shows that V_α is an increasing function on $[a, b]$

Theorem 21. If α is of bounded variation on $[a, b]$ and $V = V_\alpha$, then $D = V - \alpha$ is an increasing function on $[a, b]$ so that $\alpha = V - D$, a difference of two increasing functions.

Proof. For $a \le x \le y \le b$ we have

$$D(y) - D(x) = V(y) - V(x) - (\alpha(y) - \alpha(x)) = V_\alpha(x, y) - (\alpha(y) - \alpha(x)) \ge 0.$$

QED

We now show that the points of continuity of α are the same as the points of continuity of V_α.

Theorem 22. Let α be of bounded variation on $[a, b]$ and let $a \le c \le b$. Then α is continuous at the point $c \iff V = V_\alpha$ is continuous at c.

Proof.($\Rightarrow$) Let $\epsilon > 0$ be given and choose a partition $P = \{c = x_0 < x_1 < \cdots < x_n = b\}$ so that

$$V_\alpha(c, b) - \frac{\epsilon}{2} < A(P) \quad \text{and} \quad |\alpha(x_1) - \alpha(c)| < \frac{\epsilon}{2}.$$

Then

$$V_\alpha(c, b) - \frac{\epsilon}{2} < |\Delta\alpha_1| + \sum_{k=2}^{n} |\Delta\alpha_k| \le \frac{\epsilon}{2} + V_\alpha(x_1, b)$$

which implies $V(x) - V(c) \le V(x_1) - V(c) = V_\alpha(c, x_1) = V_\alpha(c, b) - V_\alpha(x_1, b) < \epsilon$ when $c \le x \le x_1$. Hence V is right continuous at c. A similar argument can be used for left continuity; the details are left to the reader.

($\Leftarrow$) If $a \le c < x \le b$, then $|\alpha(x) - \alpha(c)| \le V(x) - V(c)$. This implies that

$$|\alpha(c+) - \alpha(c)| \le V(c+) - V(c) = 0.$$

Similarly, $a < c \implies |\alpha(c) - \alpha(c-)| \le V(c) - V(c-) = 0$. **QED**

Corollary. If α is continuous and of bounded variation on $[a, b]$, then α is the difference of two continuous increasing functions on $[a, b]$.

Theorem23. If α is of bounded variation on $[a, b]$ and f is bounded on $[a, b]$, then

$$f \in \mathcal{R}(\alpha, a, b) \implies f \in \mathcal{R}(V, a, b),$$

where $V = V_\alpha$.

Proof. By hypothesis, we have $|f(x)| \le M$ for $a \le x \le b$. Let $\epsilon > 0$ be given and choose a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ so that

$$V(b) < \sum_{k=1}^{n} |\Delta\alpha_k| + \frac{\epsilon}{4M} \quad \text{and} \quad \sum_{k=1}^{n} |f(t_k) - f(t'_k)||\Delta\alpha_k| < \frac{\epsilon}{4}$$

for any choice of $t_k, t'_k \in [x_{k-1}, x_k]$. We now choose the $t'_k \le t_k$ so that

$$\sum_{k=1}^{n} (M_k(f) - m_k(f))|\Delta\alpha_k| \le \sum_{k=1}^{n} (f(t_k) - f(t'_k))|\Delta\alpha_k| + \frac{\epsilon}{4}.$$

Here we have used the fact that $\sum |\Delta\alpha_k| \le V(b)$. Now

$$\begin{aligned}
U(P,f,V) - L(P,f,V) &= \sum_{k=1}^{n} (M_k(f) - m_k(f))\Delta V_k \\
&= \sum_{k=1}^{n} (M_k(f) - m_k(f))(\Delta V_k - |\Delta\alpha_k|) + \sum_{k=1}^{n} (M_k(f) - m_k(f))|\Delta\alpha_k| \\
&< 2M \sum_{k=1}^{n} (\Delta V_k - |\Delta\alpha_k|) + \frac{\epsilon}{2} \\
&= 2M(V(b) - \sum_{k=1}^{n} |\Delta\alpha_k|) + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

QED

Since $D = V - \alpha$, the hypotheses of the theorem implies that $f \in \mathcal{R}(D, a, b)$ and

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, dV(x) - \int_a^b f(x)\, dD(x).$$

It follows that Theorems 16, 17, 18 and 19 extend to the case α is of bounded variation. In particular, $f \in \mathcal{R}(a, b)$ if f is of bounded variation on $[a, b]$. The details are left to the reader.

Exercise 2. Let f, α be functions on$[a, b]$ with f bounded and α of bounded variation. If $f \in \mathcal{R}(\alpha, a, b)$ show that $|f| \in \mathcal{R}(V_\alpha, a, b)$ and

$$\left| \int_a^b f(x)\, d\alpha(x) \right| \le \int_a^b |f(x)|\, dV_\alpha(x).$$

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