MATH 255: Lecture 7

The Riemann-Stieltjes Integral: Comparison and Existence Theorems

Theorem 14. Let α be an increasing function on $[a,b]$. If $f, g \in \mathcal{R}$ on $[a,b]$ and $f(x) \le g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,d\alpha(x) \le \int_a^b g(x)\,d\alpha(x).$$

Proof. For every tagged partition (P,t) of $[a,b]$ we have

$$S(P,t,f,\alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k \le \sum_{k=1}^{n} g(t_k)\Delta\alpha_k$$

since $\Delta\alpha_k \ge 0$. **QED**

It follows that $g \in \mathcal{R}(\alpha,a,b) \Rightarrow \int_a^b g(x)\,d\alpha(x) \ge 0$ if $g(x) \ge 0$ on $[a,b]$ and α is increasing.

Theorem 15. Let α be an increasing function on $[a,b]$. If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and f is bounded on $[a,b]$, then $|f| \in \mathcal{R}(\alpha)$ on $[a,b]$ and

$$\left|\int_a^b f(x)\,d\alpha(x)\right| \le \int_a^b |f(x)|\,d\alpha(x).$$

Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and let

$$m_k(f) = \inf_{x\in[x_{k-1},x_k]} f(x), \quad M_k(f) = \sup_{x\in[x_{k-1},x_k]} f(x).$$

Since $||f(x)| - |f(y)|| \le |f(x) - f(y)|$ we have $M_k(|f|) - m_k(|f|) \le M_k(f) - m_k(f)$ from which

$$U(P,|f|,\alpha) - L(P,|f|,\alpha) \le U(P,f,\alpha) - L(P,f,\alpha)$$

which implies $|f| \in \mathcal{R}(\alpha,a,b)$. The second assertion follows form the fact that $|S(P,f,\alpha)| \le S(P,|f|,\alpha)$. **QED**

Exercise 1. Prove that the converse of this theorem is false.

Theorem 16. Let α, f be functions on $[a,b]$ with α increasing and f bounded. Then

$$f \in \mathcal{R}(\alpha,a,b)) \Rightarrow f^2 \in \mathcal{R}(\alpha,a,b)).$$

Proof. We have $m_k(f^2) = m_k(|f|)^2$, $M_k(f^2) = M_k(|f|)^2$. If $|f(x)| \le M$ on $[a,b]$, we have

$$M_k(f^2) - m_k(f^2) = (M_k(|f|) + m_k(|f|))(M_k(|f|) - m_k(|f|)) \le 2M(M_k(f) - m_k(f)),$$

which implies that

$$U(P,f^2,\alpha) - L(P,f^2,\alpha) \le 2M(U(P,|f|,\alpha) - L(P,|f|,\alpha)).$$

QED

Theorem 17. Let α, f, g be functions on $[a,b]$ with α increasing and f, g bounded. Then

$$f, g \in \mathcal{R}(\alpha,a,b) \Rightarrow fg \in \mathcal{R}(\alpha,a,b).$$

Proof. $f(x)g(x) = \frac{1}{2}((f(x)+g(x))^2 - f(x)^2 - g(x)^2)$. **QED**

Theorem 18. Let α be an increasing function on $[a,b]$ and let $f \in \mathcal{R}(\alpha,a,b)$. Assume that f,g are bounded on $[a,b]$. If

$$F(x) = \int_a^x f(t)\,d\alpha(t), \quad G(x) = \int_a^x g(t)\,d\alpha(t),$$

then $f \in \mathcal{R}(G,a,b)$, $g \in \mathcal{R}(F,a,b)$ and

$$\int_a^b f(x)g(x)\,d\alpha(x) = \int_a^b f(x)\,dG(x) = \int_a^b g(x)\,dF(x).$$

Proof. For any partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ of $[a,b]$ we have

$$S(P,t,f,G) = \sum_{k=1}^n f(t_k) \int_{x_{k-1}}^{x_k} g(t)\,d\alpha(t) = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(t_k)g(t)\,d\alpha(t),$$

$$\int_a^b f(x)g(x)\,d\alpha(x) = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(t)g(t)\,d\alpha(t).$$

Therefore, if $|g(x)| \le M$ on $[a,b]$, we have

$$\begin{aligned}
|S(P,t,f,G) - \int_a^b fg\,d\alpha| &= \left|\sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(t_k) - f(t))g(t)\,d\alpha(t)\right| \\
&\le M \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f(t_k) - f(t)|\,d\alpha(t) \\
&\le M \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (M_k(f) - m_k(f))\,d\alpha(t) = M(U(P,f,\alpha) - L(P,f,\alpha))
\end{aligned}$$

which implies that $f \in \mathcal{R}(G,a,b)$ and $\int_a^b fg\,d\alpha = \int_a^b f\,dG$. The second assertion follows by interchanging f and g.

QED

We now give a sufficient condition for the existence of the Riemann-Stieltjes integral.

Theorem 19. If f is continuous on $[a,b]$ and α is increasing on $[a,b]$ then $f \in \mathcal{R}^*(\alpha)$ on $[a,b]$.

Proof. It suffices to consider the case $A = \alpha(b) - \alpha(a) > 0$. Let $\epsilon > 0$ be given and choose $\delta > 0$ so that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon/2A$ which is possible because of the uniform continuity of f; see Lecture 26. If P is any partition of norm $< \delta$, we have $M_k(f) - m_k(f) \le \epsilon/2A$ which implies that

$$U(P,f,\alpha) - L(P,f,\alpha) \le \frac{\epsilon}{2A} \sum_{k=1}^n \Delta\alpha_k = \frac{\epsilon}{2} < \epsilon.$$

This shows that f is strictly integrable with respect to α. **QED**

Corollary 1. If f is increasing on $[a,b]$ and α is continuous on $[a,b]$ then $f \in \mathcal{R}^*(\alpha)$ on $[a,b]$.

Corollary 2. If f is continuous on $[a,b]$ or if f is increasing on $[a,b]$ then $f \in \mathcal{R}$ on $[a,b]$.

Note that an increasing function on $[a,b]$ has at most a countable number of discontinuities since the number of jumps $\ge 1/n$ is finite.

A function f on $[a, b]$ is said to be **piecewise continuous** if there is a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a, b]$ such that

(a) the restriction of f to (x_{k-1}, x_k) is continuous for $1 \leq k \leq n$;

(b) $f(x_k+)$ exists for $0 \leq k < n$;

(c) $f(x_k-)$ exits for $1 \leq k \leq n$.

Exercise 2. If f is piecewise continuous on $[a, b]$, show that $f \in \mathcal{R}$ on $[a, b]$.

Theorem 8 can be extended to the case α is piecewise smooth. Recall that a function α is **piecewise smooth** if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ such that the restriction α_k of α to $[x_{k-1}, x_k]$ has a continuous derivative α_k' for $1 \leq k \leq n$. Thus $\alpha'(x)$ exists except possibly at the points $x_1, x_2, \ldots, x_{n-1}$. At these points x_k, we define $\alpha'(x_k)$ to be the average of the left-hand and right-hand limits of $\alpha'(x)$. Then

$$\int_{x_{k-1}}^{x_k} f(x)d\alpha(x) = \int_{x_{k-1}}^{x_k} f(x)d\alpha_k(x) = \int_{x_{k-1}}^{x_k} f(x)\alpha_k'(x)\,dx = \int_{x_{k-1}}^{x_k} f(x)\alpha'\,dx$$

since the the Riemann integrability and integral of a function is unchanged if we change the value of a function on a finite set of points. By additivity, we get

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx.$$

In particular, we have

$$\int_a^b \alpha'(x)\,dx = \alpha(b) - \alpha(a)$$

if α is piecewise smooth on $[a, b]$.

Exercise 3. Let f, α be functions on $[a, b]$ and suppose that α is increasing on $[a, b]$. If $a < c < b$ and f and g are both discontinuous from the right or the left at c, show that $\int_a^b f(x)\,d\alpha(x)$ cannot exist.

(Last modified 11:30 pm, January 28, 2003)