MATH 255: Lecture 6

The Riemann Integral: Strict Integrability

In this lecture we shall show that, for the Riemann integral, integrability implies strict integrability. We first show that a function which is Riemann integrable is bounded. More generally, we have the following result:

Theorem 12. If α is strictly increasing on $[a, b]$, then $f \in \mathcal{R}(\alpha, a, b) \Rightarrow f$ is bounded on $[a, b]$.

Proof. By the Cauchy Criterion, there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that

$$|\sum_{k=1}^{n}(f(t_k) - f(t'_k))\Delta\alpha_k| < 1$$

for any choice of tags t, t' for P. Since the above is true even if we interchange t_k and t'_k, we have

$$\sum_{k=1}^{n}|f(t_k) - f(t'_k)|\Delta\alpha_k < 1.$$

Hence $|f(t_k) - f(t'_k)|\Delta\alpha_k < 1$ for each k. Fixing t'_k, this shows that f is bounded on $[x_{k-1}, x_k]$ since $\Delta\alpha_k > 0$. This implies that f is bounded on $[a, b]$. **QED**

Theorem 13. $f \in \mathcal{R}(a, b) \Rightarrow f \in \mathcal{R}^*(a, b)$

Proof. Choose M so that $|f(x)| \leq M$ on $[a, b]$ and let $A = \int_a^b f(x)\,dx$. Let $\epsilon > 0$ be given and choose a partition $Q = \{a = y_0 < y_1 < \cdots < y_m = b\}$ of $[a, b]$ such that

$$|S(Q, s, f) - S(Q, s', f)| = \sum_{i=1}^{m}(f(s_i) - f(s'_i))\Delta y_i < \frac{\epsilon}{3}$$

for any choice of tags s, s' for Q. If P is any partition $a = x_0 < x_1 < \cdots < x_n = b$, we can choose tags t, t' for P so that $f(t'_k) \leq f(t_k)$ and

$$U(P, f) - L(P, f) < \sum_{k=1}^{n}(f(t_k) - f(t'_k))\Delta x_k + \frac{\epsilon}{3}.$$

Let S_1 is the sum of the terms $(f(t_k) - f(t'_k))\Delta x_k$ for which the interval (x_{k-1}, x_k) does not meet Q. For these terms we have $[x_{k-1}, x_k] \subseteq [y_{i-1}, y_i]$ for some i which implies that

$$(f(t_k) - f(t'_k))\Delta x_k \leq (f(t_k) - f(t'_k))\Delta y_i.$$

If we set $s_i = t_k$, $s'_i = t'_k$ for these i and arbitrarily otherwise, we have

$$S_1 \leq \sum_{i=1}^{m}(f(s_i) - f(s'_i))\Delta y_i < \frac{\epsilon}{3}.$$

If S_2 is the sum of the remaining terms, we have $S_2 \leq 2mM||P|| < \epsilon/3$ if $||P|| < \delta = \epsilon/6mM$. Thus $U(P, f) - L(P, f) < \epsilon$ if $||P|| < \delta$ and hence $|A - S(P, t, f)| < \epsilon$ if $||P|| < \delta$. **QED**

Exercise. If $f \in \mathcal{R}$ on $[a, b]$ and g is a function on $[a, b]$ such that $f(x) = g(x)$ for all but a finite number of $x \in [a, b]$, show that $g \in \mathcal{R}$ on $[a, b]$ and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

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