MATH 255: Lecture 28

Normed spaces

Let V be a real or complex vector space. A function $|| \ ||$ on V with values in $\mathbb{R}$ is called a norm on V if

(N1) For any $v \in V$, we have $||v|| \geq 0$ with equality $\iff v = 0$;

(N2) For any scalar, $v \in V$, we have $||cv|| = |c|||c||$;

(N3) For any $u, v \in V$, we have $||u + v|| \leq ||u + v||$.

Any norm on V defines a distance function $d(u, v) = ||u - v||$ so that any normed space is a metric space. A normed vector space is said to be complete if it is complete with respect to this metric. A complete normed space is called a Banach space.

Example 1. The set $V = B(X)$ (respectively $B_{\mathbb{C}}(X)$), of bounded real-valued (respectively, complex-valued) functions on a set X is a vector space under the usual operations of pointwise addition and multiplication by scalars. If we define

$$||f||_\infty = \sup_{x \in X} |f(x)|,$$

then $|| \ ||_\infty$ is a norm on V. With this norm, V is complete. If $X = \mathbb{N}$, this space is denoted by ℓ^∞ (respectively $\ell^\infty_{\mathbb{C}}$).

Example 2. If V is an inner product space with inner product $< \ , \ >$, then $||v|| = \sqrt{< v, v >}$ defines an inner product on V. If V is complete, then V is called a Hilbert space. For example, $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces.

Example 3. If $p > 0$, let $V = \ell^1$ (resp. $\ell^1_{\mathbb{C}}$) the vector space of infinite sequences $u = (x_n)_{n \geq 0}$ of real (resp. complex) scalars x_n such that the series $\sum_{n=0}^\infty |x_n|$ is convergent. Then

$$||u|| = \sqrt{\sum_{n=0}^\infty |x_n|}$$

is a norm on V. To show that V is complete with this norm, let (u_n) be a Cauchy sequence in V with $u_n = (x_{nk})_{k \geq 0}$ be a Cauchy sequence. Since

$$|x_{nk} - x_{mk}| \leq \sum_{k=0}^\infty |x_{nk} - x_{mk}| = ||u_n - u_m||_1$$

we see that $x_k = \lim_{n \to \infty} x_{nk}$ exists. Since

$$\sum_{k=0}^N |x_{nk} - x_{mk}| \leq ||u_n - u_m||,$$

we obtain, on passing to the limit with respect to m,

$$\sum_{k=0}^N |(x_{nk} - x_k)| \leq ||u_n - u_m||.$$

Passing to the limit in N, we get

$$\sum_{k=0}^\infty |x_{nk} - x_k| \leq ||u_n - u_m||$$

which shows that $u_n - u \in V$ and hence that $u \in V$ and, moreover, that u_n converges to u.

Exercise 1. Let $V = \ell^2$ (resp. $\ell^2_{\mathbb{C}}$) the vector space of infinite sequences $x = (x_n)_{n \geq 0}$ of real (resp. complex) scalars x_n such that the series $\sum n = 0^\infty |x_n|$ is convergent. Show that

$$< x, y >= \sqrt{\sum_{n=0}^{\infty} |x_n y_n|}$$

is an inner product on V and that V is a Hilbert space.

A series $\sum_{n=1}^\infty$ in a normed space is said to converge if the partial sums $s_n = u_1 + \cdots + u_n$ converge. The series is said to converge absolutely if the series $\sum_{n=1}^\infty ||u_n||$ converges.

Theorem 10. A normed space V is complete if and only if every Cauchy sequence converges.

Proof. If $\sum_{n=1}^\infty u_n$ is a series in V and s_n is the n-th partial sum, we have for $n > m$

$$||s_n - s_m|| = ||u_{m+1} + \ldots + u_n|| \leq ||u_{m+1} + \cdots + ||u_n||$$

which shows that the partial sums form a Cauchy sequence in V if the series converges absolutely. Hence, if V is complete, every absolutely convergent series converges.

Conversely, suppose every absolutely convergent series in V is convergent and let (u_n) be a Cauchy sequence in V. To prove convergence of this sequence, we only have show that it has a convergent subsequence. After passing to a subsequence, we may assume that $||u_n - u_{n+1}|| < 1/2^n$ for all $n \geq 1$. If we set $v_n = u_n - u_{n+1}$, we have

$$\sum_{k=1}^{n} v_k = u_1 - u_{n+1}$$

which show that that the series $\sum v_n$ converges if and only if the sequence (u_n) converges. But $\sum v_n$ is absolutely convergent since $\sum ||v_n|| < \sum 1/2^n$ and hence is convergent. **QED**

Two norms $|| \ ||_1$ and $|| \ ||_2$ on V are said to be equivalent if there are constants $C_1, C_2 > 0$ such that

$$||u||_1 \leq C_1 ||u||_2 \quad \text{and} \quad ||u||_2 \leq C_2 ||u||_1.$$

For example, if $u = (x_1, \ldots, x_n) \in \mathbb{R}^n($ or $\mathbb{C}^n)$, then

$$||u||_\infty = \max_{1 \leq k \leq n} k|x_k|, \quad ||u||_1 = \sum_{k=1}^{n} |x_k|, \quad ||u||_2 = \sqrt{\sum_{k=1}^{n} x_n^2}$$

are equivalent norms. Equivalent norms define the same topology for V.

Exercise 3. If A is an $n \times n$ matrix over $\mathbb{R}$ or $\mathbb{C}$, show that the series $\sum_{n=0}^\infty A^n/n!$ is absolutely convergent for the ∞-norm on the space of $n \times n$ matrices. This matrix is denoted by $\exp(A)$ or e^A. Hint: use the fact $||AB||_1 \leq ||A||_1 ||B|_1$.

Exercise 4. If $AB = BA$, show that $\exp(A + B) = \exp(A)\exp(B)$. Deduce that $\exp(A)$ is invertible with inverse $\exp(-A)$.

If f is a function defined on an interval $I \subseteq \mathbb{R}$ with values in a normed space V, one can define the derivative $f'(a)$ at a point $a \in V$ as the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. If f, g have a derivative at a, then so does $h = cf + dg$ and $h'(a) = cf'(a) + dg'(a)$. If $V = \mathbb{R}^n$ and $f(t) = (f_1(t), \ldots, f_n(t))$, then $f'(a) = (f_1'(a), \ldots, f_n'(a))$. If f, g take values in the vector space of $n \times n$ matrices, and $h(t) = f(t)g(t)$ then $h'(a) = f'(a)g(a) + f(a)g'(a)$ so that the usual rules of calculus apply.

Mean-Value Theorem. Let f be a continuous mapping from $[a,b]$ into a normed space V. Suppose that $f'(t)$ exists and $||f'(t)|| \leq M$ for all $t \in (a,b)$. Then $||f(b) - f(a)|| \leq M(b-a)$.

Proof. Let $\epsilon > 0$ be given and let I be the set of points $x \in [a,b]$ such that

$$||f(x) - f(a)|| \leq (M+\epsilon)(x-a) + \epsilon.$$

Let $c = \sup I$. Then $c > a$ since f is continuous at a. Also, by the continuity of f we have

$$||f(c) - f(a)|| \leq (M+\epsilon)(c-a) + \epsilon.$$

Suppose that $c < b$. Then, since $||f'(c)|| \leq M$, we have $||f(c+h) - f(c)|| \leq (M+\epsilon)h$ for some $h > 0$. But then

$$\begin{aligned} ||f(c+h) - f(a)|| &\leq ||f(c+h) - f(c)|| + ||f(c) - f(a)|| \\ &\leq (M+\epsilon)h + (M+\epsilon)(c-a) + \epsilon \\ &= (M+\epsilon)(c+h-a) + \epsilon \end{aligned}$$

which is a contradiction. **QED**

Corollary. If f is continuous on $[a,b]$ and $f'(t) = 0$ on (a,b), then $f = 0$ on $[a,b]$.

One can also extend the theory of Riemann integration of functions on an interval $[a,b]$ with values in a normed space V. If $V = \mathbb{R}^n$ and $f(t) = (f_1(t), \ldots, f_n(t))$, then f is Riemann integrable if and only if each f_i is Riemann integrable and

$$\int_a^b f(t)\,dt = \Big(\int_a^b f_1(t)\,dt, \ldots, \int_a^b f_n(t)\,dt\Big).$$

Moreover, $||\int_a^b f(t)\,dt||_1 \leq \int_a^b ||f(t)||_1\,dt$. It follows that one can integrate, term by term, uniformly convergent series of continuous functions on $[a,b]$ with values in $\mathbb{R}^n$.

Exercise 5. If $f(t) = \exp(tA)$, show that $f'(t) = Af(t)$.