MATH 255: Lecture 27

The Topology of Metric Spaces

If (S, d) is a metric space, we let $\mathcal{T} = \mathcal{T}_S$ be the set of open sets of the metric space. The set $\mathcal{T}$ is a collection of subsets of S that has the following properties:

(O1) If $U_i \in \mathcal{T}$ for $i \in I$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$;

(O2) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$;

(O3) $\emptyset,\ S \in \mathcal{T}$.

A collection $\mathcal{T}$ of subsets of a set S that satisfies these three properties is called a topology on S and the members of $\mathcal{T}$ are called open sets. The pair $(S, \mathcal{T})$ is called a topological space. If $(S, \mathcal{T})$ and $(S, \mathcal{T}')$ are topological spaces, a mapping $f : S \to S'$ is said to be continuous if the inverse image of an open set is open. The topological spaces are said to be isomorphic if f is bijective and both f and f^{-1} are continuous. A bijective continuous mapping is also called a homeomorphism.

Exercise 1. If $(S, \mathcal{T})$ is a topological space and X is a subset of S, show that the set

$$\mathcal{T}_X = \{X \cap U \mid U \in \mathcal{T}\}$$

is a topology for X. With this induced topology, X is a called a subspace of X.

A property of metric spaces that can be described entirely in terms of open sets is said to be topological. For example, continuity is a topological property while boundedness is not. Two metrics for a set S are said to be equivalent if the associated topologies are the same.

Theorem 8. If d, d' are metrics on S, then the following are equivalent:

(a) The metrics d and d' are equivalent;

(b) The metrics D and d' determine the same convergent sequences;

(c) $(\forall p \in S)(\forall \epsilon > 0)(\exists \delta > 0)$ such that $D_\delta(p) \subseteq D'_\epsilon(p)$ and $D_\delta(p)' \subseteq D_\epsilon(p)$.

The proof is left as an exercise. Two metrics d and d' on a set S are said to be strongly equivalent if there are constants $C, C' > 0$ such that $d(x, y) \le Cd'(x, y)$ and $d'(x, y) \le C'd(x, y)$. For example, the Euclidean metric d_2 and the uniform metric d_∞ are strongly equivalent since

$$d_\infty(x, y) \le d_2(x, y) \quad \text{and} \quad d_2(x, y) \le \sqrt{n} d_\infty(x, y).$$

Strongly equivalent metrics are equivalent.

Exercise 2. If d is a metric, show that $d'(x, y) = \dfrac{d(x, y)}{1 + d(x, y))}$ is an equivalent metric.

If (S_1, d_1) and (S_2, d_2) are metric spaces, then

$$d(x, y) = \max(d_1(x, y), d_2(x, y))$$

is a metric on $S_1 \times S_2$. The set $S_1 \times S_2$ with this metric is called the Cartesian product of the metric spaces (S_1, d_1) and (S_2, d_2). The metric $d(x, y) = d_1(x, y) + d_2(x, y)$ is an equivalent metric. We have $(x_n, y_n) \to (x, y)$ in $S_1 \times S_2$ if and only if $x_n \to x$ in S_1 and $y_n \to y$ in S_2.

Exercise 3. If $f_1 \to S_1$ and $f_2 \to S_2$ are continuous, show that $f \to S_1 \times S_2$, where $f(p) = (f_1(p), f_2(p))$ is continuous.

Exercise 4. If f, g are continuous real valued functions on a metric space S and $c \in \mathbb{R}$, prove that the functions $f + g$, fg, cf defined by $(f + g)(p) = f(p) + g(p)$, $fg(p) = f(p)g(p)$, $(cf)(p) = c(f(p))$ are continuous.

Another topological property is that of connectiveness. A topological space is said to be connected if it cannot be expressed as the union of two non-empty disjoint open sets. A subset of a topological space is said to be connected if it is connected as a subspace. The connected subsets of $\mathbb{R}$ are the intervals.

Theorem 9. The continuous image of a connected set is connected.

The proof is left to the reader.

Theorem 10. If $(C_i)_{i \in I}$ is a family of connected subsets of a space S and C is a connected subset of S which has a non-empty intersection with each C_i then $X = C \cup \bigcup_{i \in I} C_i$ is connected.

Proof. Suppose that X is not connected and let $X = U \cup V$ with U, V open in X, disjoint and non-empty. Since C is connected and $C = (C \cap U) \cup (C \cap V)$ we must have $C \cap U = \emptyset$ or $C \cap V = \emptyset$. Hence $C \subseteq U$ or $C \subseteq V$. Say $C \subseteq U$. Similarly, each C_i is a subset of U or V. But $C_i \cap C \neq \emptyset$ implies $C_i \subseteq U$. Hence $X \subseteq U$, which is a contradiction.