MATH 255: Lecture 26

Introduction to Metric Spaces: Continuity

Let (S, d), (S', d') be metric spaces and let $f : S \to S'$ be a function which maps S into S'. If $a \in S$ and $L \in S'$, then

$$L = \lim_{x \to a} f(x) \iff (\forall \epsilon > 0)(\exists \delta > 0)\ 0 < d(x, a) < \delta \implies d(f(x), f(a)) < \epsilon.$$

Then f is said to be continuous at $a \in S$ if $L = f(a)$. This is equivalent to

$$(\forall \epsilon > 0)(\exists \delta > 0)\ f(D_\delta(a)) \subseteq D_\epsilon(f(a)).$$

If we define a neighbourhood of a point p in a metric space to be any set containing a in its interior, then f is continuous at a if and only if the inverse image of any neighbourhood of $f(a)$ is a neighbourhood of a. It follows that f is continuous on S if and only if the inverse image of an open set in S' is open in S.

Exercise 1. Prove that the composition of continuous functions is continuous.

The mapping $f : S \to S'$ is said to be open if the image of an open set is open. A continuous map need not be open. For example $f(x) = x^2$ is a continuous mapping of $\mathbb{R}$ int $\mathbb{R}$ but $f((-1, 1)) = [0, 1)$ is not open.

Theorem 4. The continuous image of a compact set is compact.

Proof. Let $f : S \to S'$ be continuous and let X be a compact subset of S. If $(G_i)_{i\in I}$ is an open covering of $f(X)$ then $(f^{-1}(G_i))_{i\in I}$ is an open covering of X since $X \subseteq f^{-1}(f(X)$. If $(f^{-1}(G_i))_{i\in J}$ is a finite subcovering, then $(G_i)_{\in J}$ is a finite subcovering of $f(X)$ since $f(f^{-1}(Y)) = Y$. **QED**

Corollary. If $f : S \to \mathbb{R}$ is continuous and X is a compact subset of S, then f has a maximum and minimum on X.

A function $f : S \to S'$ is said to be uniformly continuous if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in S)\ d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon.$$

Theorem 5. If $f : S \to S'$ is continuous and S is compact, then f is uniformly continuous.

Proof. Let $\epsilon > 0$ be given. For each $p \in S$ there exist $\delta_p > 0$ such that $f(D_{\delta_p}) \subseteq D_{\epsilon/2}(f(p))$. Since S is compact, there are points $p_1, \ldots, p_n$ such that $S = \bigcup_{i=1}^n D_{\delta_i}(p_i)$, where $\delta_i = \delta_{p_i}/2$. Let $\delta = \min(\delta_1, \ldots, \delta_n)$. If $d(x, y) < \delta$, then $d(x, p_i) < \delta_i$ for some i and so

$$d(y, p_i) < d(y, x) + d(x, p_i) < 2\delta_i = \delta_{p_i}$$

so that $x, y \in D_{\delta_{p_i}}(p_i)$ which implies that $d(f(x), f(p_i)) < \epsilon/2$ and $d(f(x), f(p_i)) < \epsilon/2$. Hence $d(f(x), f(y)) < \epsilon$. **QED**

Corollary. If f is a continuous function on $\mathbb{R}^n$ and X is a closed and bounded subset of $\mathbb{R}^n$, then f is uniformly continuous on X.

We now give some applications of this result to the theory of integration.

Theorem 6. If f is a continuous real-valued function on $[a, b] \times [c, d]$, then

$$F(x) = \int_c^d f(x, y)\, dy$$

is continuous on $[a, b]$. If in addition, $f_x = \frac{\partial f}{\partial x}$ is continuous on $[a, b] \times [c, d]$, we have

$$F'(x) = \int_c^d \frac{\partial f}{\partial x}(x, y)\, dy.$$

Proof. Let $\epsilon > 0$ be given. By the uniform continuity of f, there exists $\delta > 0$ such that

$$|s - t| < \delta \implies |f(s, y) - f(t, y)| < \epsilon/(d - c).$$

We then have

$$|F(s) - F(t)| \leq \int_c^d |f(s, y) - f(t, y)|\, dy < \epsilon$$

which yields the uniform continuity of F on $[a, b]$.

Similarly, by the uniform continuity of f_x,

$$(\exists \delta > 0)\ |s - t| < \delta \implies |f_x(s, y) - f_x(t, y)| < \epsilon/(d - c).$$

Since $f(x + h, y) - f(x, y) = hf_x(x + \theta h, y)$ with $0 < \theta < 1$ we have, for $|h| < \delta$,

$$\left|\frac{F(x + h) - F(x)}{h} - \int_c^d f_x(x, y)\, dy\right| \leq \int_c^d |f(x + \theta h, y) - f(x, y)|\, dy < \epsilon.$$

QED

We now extend this result to improper integrals.

Theorem 7. Suppose that f is continuous on $[a, b] \times [c, \infty]$ and that

$$F(x) = \int_c^\infty f(x, y)\, dy$$

is uniformly convergent for $x \in [a, b]$. Then F is continuous on $[a, b]$. If, in addition, f_x is continuous on $[a, b] \times [c, \infty]$ and $\int_c^\infty f_x(x, y)\, dy$ is uniformly convergent for $x \in [a, b]$, then

$$F'(x) = \int_c^\infty f_x(x, y)\, dy.$$

Proof. Let $\epsilon > 0$ be given and choose d so that $|\int_d^\infty f(x, y)\, dy| < \epsilon/3$. Now choose $\delta > 0$ so that $|s - t| < \delta \implies |f(s, y) - f(t, y)|\epsilon/3(d - c)$ for all $s, t \in [a, b]$, $y \in [c, d]$. Then, for $|s - t| < \delta$,

$$|F(s) - F(t)| \leq \int_c^d |f(s, y) - f(t, y)|\, dy + 2\epsilon/3 < \epsilon.$$

Similarly, if $\int_c^\infty f_x(x, y)\, dy$ is uniformly convergent for $x \in [a, b]$, we can choose d so that

$$|\int_d^\infty f_x(x, y)\, dy| < \epsilon/3$$

for all $x \in [a, b]$. Then, by the uniform continuity of f_x, we can choose $\delta > 0$ so that

$$|s - t| < \delta \implies |f_x(s, y) - f_x(t, y)| < \epsilon/3(d - c).$$

Since $f(x + h, y) - f(x, y) = hf_x(x + \theta h, y)$ with $0 < \theta < 1$ we have, for $|h| < \delta$,

$$\left|\frac{F(x + h) - F(x)}{h} - \int_c^\infty f_x(x, y)\, dy\right| \leq \int_c^d |f_x(x + \theta h, y) - f_x(x, y)|\, dy + 2\epsilon/3 < \epsilon.$$

QED

Example. By Dirichlet's test, $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$ converges. It follows, by Abel's test, that

$$F(y) = \int_0^\infty e^{-xy} \frac{\sin x}{x}\, dx$$

converges uniformly for $y \in [0, \infty)$. Hence F is continuous on $[0, \infty)$. Now, for $y > 0$, we have

$$F'(y) = -\int_0^\infty e^{-xy} \sin x \, dx$$

since this integral converges uniformly for $y \in [\epsilon, \infty)$ for any $\epsilon > 0$ by the Weierstrass M-test. By elementary calculus,

$$\int_0^a e^{-xy} \sin x \, dx = \frac{e^{-ay}(-y \sin a - \cos a)}{1 + y^2} + \frac{1}{1 + y^2}.$$

If $y > 0$ and we let $a \to \infty$, we get

$$\int_0^\infty e^{-xy} \sin x \, dx = \frac{1}{1 + y^2},$$

so that $F'(y) = -1/(y^2 + 1)$ for $y > 0$. It follows that

$$F(y) - F(b) = -\int_b^y \frac{dt}{1 + t^2} = \tan^{-1} b - \tan^{-1} y$$

for $y, b > 0$. Since $F(b) \to 0$ and $\tan^{-1} b \to \pi/2$ as $b \to \infty$, we get

$$\int_0^\infty e^{-xy} \frac{\sin x}{x} \, dx = \frac{\pi}{2} - \tan^{-1} y$$

for $y > 0$. Since both sides are continuous for $y \geq 0$, the equation holds for $y = 0$ which gives

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$

A sequence of functions (f_n) on a set X with values in a metric space S is said to converge pointwise to a function f on X with values in S if for all $x \in X$ we have $\lim f_n(x) = f(x)$. The sequence is said to converge uniformly to f if

$$(\forall \epsilon > 0)(\exists N)(\forall n \geq N)(\forall x \in X) \; d(f_n(x), f(x)) < \epsilon.$$

Such a sequence satisfies the uniform Cauchy condition

$$(\forall \epsilon > 0)(\exists N)(\forall m, n \geq N)(\forall x \in X) \; d(f_m(x), f_n(x)) < \epsilon.$$

If S is complete and (f_n) satisfies the uniform Cauchy condition then (f_n) converges uniformly to a function f on X with values in S.

Exercise 2. Let (f_n) be a uniformly convergent sequence of continuous functions on a metric space X with values in a metric space S. Show that the limit function f is continuous.

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