MATH 255: Lecture 23

Improper Integrals

Let α be an increasing function on $[a, b)$ with $a < b$ and b possibly ∞. If f is a function on $[a, b)$, which is unbounded if b is finite, such that $f \in \mathcal{R}(\alpha, a, c)$ for every $a < c < b$ and

$$\lim_{c \to b} \int_a^c f(x)\, d\alpha(x)$$

exists, we define $\int_a^b f(x)\, d\alpha(x)$ to be this limit. Such an integral is called a convergent improper integral. If the limit does not exist, it is called a divergent improper integral. If f is positive, $F(c) = \int_a^c f(x)\, d\alpha(x)$ is an increasing function of c and hence the limit exits if and only if $F(c)$ is bounded for $a \le c < b$. Similarly, one can define an improper integral with respect to the lower limit a with a possibly $-\infty$. If the integral is improper with respect to both limits, we can write it as a sum of two improper integrals of the above type and the integral is said to be convergent if and only if both of these improper integrals are convergent.

Example 1. If $0 < p \neq 1$, the improper integral

$$\int_0^1 \frac{dx}{x^p} = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{dx}{x^p} = \lim_{\epsilon \to 0} \frac{x^{1-p}}{1-p}\bigg|_\epsilon^1 = \frac{1}{1-p} - \lim_{\epsilon \to 0} \frac{\epsilon^{1-p}}{1-p}$$

which converges to $1/(1-p)$ if $p < 1$ and diverges if $p > 1$. The reader will show that the integral $\int_0^1 dx/x$ is divergent.

Example 2. If $0 < p \neq 1$, the improper integral

$$\int_1^\infty \frac{dx}{x^p} = \lim_{c \to \infty} \int_1^c \frac{dx}{x^p} = \lim_{c \to \infty} \frac{x^{1-p}}{1-p}\bigg|_1^c = \frac{1}{p-1} + \lim_{c \to \infty} \frac{c^{1-p}}{1-p}$$

which converges to $1/(p-1)$ if $p > 1$ and diverges if $p < 1$. The reader will show that the integral diverges if $p = 1$.

An improper integral $\int_a^b f\, d\alpha$ is said to converge absolutely if $\int_a^b |f|\, d\alpha$ converges. It is left to the reader to show that absolute convergence implies convergence. If $\int_a^b f\, d\alpha$ converges but not absolutely, then the convergence is said to be conditional. If f and g are positive functions and $f(x) \le Mg(x)$ then $\int_a^b g\, d\alpha$ converges implies $\int_a^b f\, d\alpha$ converges. Similarly, $M > 0$ and $f(x) \ge Mg(x) \ge 0$ then $\int_a^b g\, d\alpha$ diverges implies $\int_a^b f\, d\alpha$ diverges.

Example 3. For $x > 0$, the Gamma function is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt = \int_0^1 t^{x-1} e^{-t}\, dt + \int_1^\infty t^{x-1} e^{-t}\, dt.$$

The first integral is proper if $x \ge 1$; so we assume $0 < x < 1$. Then $0 \le t^{x-1} e^{-t} \le 1/t^{1-x}$ which implies the convergence of the first integral. For the second integral, we have $t^{x-1} e^{-t} \le 1/t^2$ for $t \ge 1$ which implies convergence.

Theorem (Dirichlet). Suppose that $|\int_a^c f(x)\, dx| \le M$ for all $c \ge A$ and that $g(x)$ is a decreasing function for $x \ge a$ with $\lim_{x \to \infty} g(x) = 0$. Then

$$\int_a^\infty f(x) g(x)\, dx \quad \text{is convergent and} \quad \left| \int_a^\infty f(x) g(x)\, dx \right| \le M g(a).$$

Proof. Let $F(x) = \int_a^x f(t)\, dt$. Then

$$\int_a^c f(x) g(x)\, dx = \int_a^c g(x)\, dF(x) = g(c)F(c) + \int_a^c F(x) d(-g(x))$$

and

$$\int_a^c |F(x)|\, d(-g(x)) \le M \int_a^c d(-g(x)) = M(g(a) - g(c)) \to Mg(a).$$

QED

Example 4. Since $|\int_a^x \sin(x)\, dx| \le 2$ and $1/x$ decreases to 0 as $x \to \infty$, we see that

$$F(s) = \int_1^\infty \frac{\sin x}{x^s}\, dx$$

converges for $s > 0$. It is left to the reader to show that the convergence is uniform in s for $s \ge \epsilon > 0$. Use this to show that $F(s)$ is continuous for $s > 0$.

Theorem (Abel). If $g(x)$ is a monotone function such that $\lim_{x\to\infty} g(x) = L \ne \pm\infty$ then

$$\int_a^\infty f(x)\, dx \quad \text{converges} \quad \implies \int_a^\infty f(x)g(x)\, dx \quad \text{converges.}$$

Moreover, $|\int_a^x f(t)\, dt| \le M$ for all $x \implies |\int_a^\infty f(x)g(x)\, dx| \le ML + M|g(a) - L|$.

The proof is left as an exercise for the reader.

Example 5. Since $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$ converges, Abel's theorem implies $\displaystyle\int_0^\infty \frac{\sin x}{x} \tan^{-1} x\, dx$ is convergent.

Example 6. The integral $\int_0^\infty \sin x^2\, dx$ is convergent since the change of variable $t = x^2$ transforms it into the integral

$$\int_0^\infty \frac{\sin t}{\sqrt{t}}\, dt$$

which is convergent by Dirichlet's theorem. Thus $\int_a^\infty f(x)\, dx$ may converge without $f(x)$ converging to 0 as $x \to \infty$. This is not the case if $f(x)$ is strictly positive and decreasing. Why?

The theory of infinite series is parallel to the theory of improper integrals but we also have a direct connection between the two theories given by the Euler summation formula. We assume that f is a positive, decreasing, continuous function on $[1, \infty]$. If $a_n = f(n)$, we have

$$\sum_{n=2}^{n} a_n = \int_1^n f(x)\, dx - \int_1^n f(x)\, d((x)),$$

where $((x)) = x - [x]$ is the fractional part of x. By the integral test,

$$\sum_{n=1}^\infty a_n \quad \text{converges} \quad \iff \int_1^\infty f(x)\, dx \quad \text{converges.}$$

Hence, in the case of convergence, $\int_1^\infty f(x)\, d((x))$ also converges and

$$\sum_{n=2}^\infty a_n = \int_1^\infty f(x)\, dx - \int_1^\infty ((x)) df(x).$$

It follows that, if $f(x)$ is piecewise smooth on $[1, n]$ for every $n > 1$, we have

$$\sum_{n=2}^\infty a_n = \int_1^\infty f(x)\, dx - \int_1^\infty ((x)) f'(x)\, dx$$

which expresses a convergent series in terms of convergent improper integrals. For example, if $f(x) = 1/x^s$, we get the important formula

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \frac{s}{s-1} - s\int_1^\infty \frac{((x))}{x^{s+1}}\, dx.$$

(Last updated 4:30 pm, March 29, 2003)