MATH 255: Lecture 22

Power Series: The Binomial Series

The Taylor series for the function $f(x) = (1+x)^\alpha$ about $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n = 1 + \alpha + \frac{\alpha(\alpha-1)}{2!}x + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots .$$

This series is called the **binomial series**. We will determine the interval of convergence of this series and when it represents $f(x)$. If α is a natural number, the binomial coefficient

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

is zero for $\alpha > n$ so that the binomial series is a polynomial of degree α which, by the binomial theorem, is equal to $(1+x)^\alpha$. In what follows we assume that α is not a natural number.

If a_n is the n-th term of the binomial series, we have

$$\frac{a_{n+1}}{a_n} = \frac{\alpha-n}{n+1}x \to -x \text{ as } n \to \infty$$

so that the radius of convergence of the binomial series is 1.

When $x = -1$, we have

$$\frac{a_{n+1}}{a_n} = \frac{n-\alpha}{n+1} \quad \text{and} \quad \lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \alpha + 1.$$

Since a_n has constant sign for $n > \alpha$, Raabe's test applies to give convergence for $\alpha > 0$ and divergence for $\alpha < 0$.

If $x = 1$, the series becomes alternating for $n > \alpha$. By Raabe's test the series converges absolutely if $\alpha > 0$. If $\alpha \le -1$ then $|a_{n+1}| \ge |a_n|$ so that the series diverges. The remaining case is $-1 < \alpha < 0$. In this case $|a_n| > |a_{n+1}|$ so that we only have to show that $a_n \to 0$. Setting $u = 1 + \alpha$, we have

$$|a_n| = \prod_{k=1}^{n}(1 - \frac{u}{n}) \implies \log|a_n| = \sum_{k=1}^{n}\log(1 - \frac{u}{n}) < -u\sum_{k=1}^{n}\frac{1}{k} \to -\infty$$

which implies that $a_n \to 0$.

Theorem (Binomial Theorem). The interval of convergence I of the binomial series is

$$[-1, 1] \text{ if } \alpha > 0, \quad (-1, 1] \text{ if } -1 < \alpha < 0, \quad (-1, 1) \text{ if } \alpha \le -1.$$

The convergence at the endpoints is absolute $\iff \alpha > 0$. On I we have

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n.$$

Proof. We only have to prove the last statement. By Taylor's theorem, we have

$$(1+x)^\alpha = \sum_{k=0}^{n-1}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + R_n(x),$$

where

$$R_n(x) = \frac{1}{(n-1)!}\int_0^x f^{(n)}(t)(x-t)^{n-t}\,dt = \frac{1}{(n-1)!}\int_0^x \alpha(\alpha-1)\cdots(\alpha-n+1)(1+t)^{\alpha-n}(x-t)^{n-1}\,dt.$$

Using the first mean value theorem for integrals, we obtain

$$R_n(x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}(1+\theta x)^{\alpha-n}(x-\theta x)^{n-1}\int_0^x dt,$$

where $0 < \theta < 1$. Simplifying, we get

$$R_n(x) = c_n(xt)\alpha x(1+\theta x)^{\alpha-1} \quad \text{with} \quad c_n(s) = \frac{(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}s^{n-1}$$

and $t = (1-\theta)/(1+\theta x)$. Then $(1+s)^{\alpha-1} = \sum_{n=1}^{\infty} c_n(s)$. Now let $x \in I$. Since $0 < t < 1$ if $x > -1$, we have $|xt| < 1$ and so the series $\sum_{n=1}^{\infty} c_n(xt)$ converges if $x > -1$. So its n-th term $c_n(xt)$ converges to zero. If $x = -1 \in I$, we have $t = 1$. Since the series for $(1+x)^\alpha$ converges for $x = -1$ we have $\alpha > 0$ and hence $\alpha - 1 > -1$. Since the series $\sum_{n=1}^{\infty} c_n(s)$ converges at $s = 1$ if $\alpha > -1$, we have $c_n(-1) \to 0$ since $|c_n(-1)| = |c_n(1)| \to 0$.

QED

Example 1. For $|x| < 1$ we have,

$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

By the binomial theorem, we have

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}x^{2n} + \cdots$$

for $|x| < 1$. Integrating, we get

$$\sin^{-1}(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{x^{2n+1}}{2n+1} + \cdots.$$

The series converges when $x = 1$ by Raabe's test since

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) = \frac{6n^2+5n}{4n^2+10n+6} \to \frac{3}{2} > 1$$

Since the series for $x = -1$ is the negative of the above series, $[-1, 1]$ is the interval of convergence of the power series. Since the series in continuous on its interval of convergence and $\sin^{-1}(x)$ is continuous there as well, we see that the power series expansion is valid on $[-1, 1]$. It follows that

$$\frac{\pi}{2} = 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{1}{2n+1} + \cdots.$$

We leave to the reader the task of proving that the remainder after n terms is less than $2/\sqrt{n+2}$ for $n \geq 10$. This would give an estimate of $n = 4000000$ to get π correct to 2 decimal places.

Since $\sin^{-1}(1/2) = \pi/6$, we also have

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\left(\frac{1}{2}\right)^3 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n+1} + \cdots$$

which converges more rapidly than then previous series. In fact, to compute π to 2 decimal places, 3 terms suffice.

Example 2. The substitution $x = \sin\theta$ reduces the improper integral

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx \quad (k^2 < 1)$$

to the integral

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\,d\theta.$$

But

$$(1-k^2\sin^2\theta)^{-1/2} = 1 + \frac{1}{2}k^2\sin^2\theta + \frac{1\cdot 3}{2\cdot 4}k^4\sin^4\theta + \cdots \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}k^{2n}\sin^{2n}\theta + \cdots$$

with $|k^2\sin^2\theta| < k^2 < 1$ so that we can integrate term by term to get

$$\begin{aligned} K &= 1\frac{\pi}{2} + \frac{1}{2}k^2\int_0^{\pi/2}\sin^2\theta\,d\theta + \frac{1\cdot 3}{2\cdot 4}k^4\int_0^{\pi/2}\sin^4\theta\,d\theta + \cdots \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}k^{2n}\int_0^{\pi/2}\sin^{2n}\theta\,d\theta + \cdots \\ &= \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\right)^2 k^{2n} + \cdots\right) \end{aligned}$$

since

$$\int_0^{\pi/2}\sin^{2n}\theta\,d\theta = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{\pi}{2}.$$

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