MATH 255: Lecture 21

Power Series

A series of functions $\sum_{n=0}^{\infty} a_n(x-c)^n$ is said to be a **powers series with center** $x = c$. Setting $s = x - c$ we get a power series $\sum_{n=0}^{\infty} a_n s^n$ with center $s = 0$. For this reason, we lose no generality in assuming that the center is zero.

If $|a_n x_0^n| \le M$ for some $x_0 \ne 0$, then

$$\sum |a_n x^n| \le \sum |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \le \sum M r^n$$

with $r = |x|/|x_0|$. This shows that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < |x_0|$ and that there is an $R > 0$, possibly $= \infty$, such that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$ and diverges for $|x| > R$. This number R is called the **radius of convergence** of the power series $\sum_{n=0}^{\infty} a_n x^n$. Thus the set of points where the power series converges is the interval $(-R, R)$ together with possibly one or more of the endpoints $\pm R$. This interval is called the **interval of convergence** of the power series.

Theorem. If R is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$, then $R = 1/\overline{\lim}\ \sqrt[n]{|a_n|}$.

Proof. If $|x| < R$, then $\sum_{n=0}^{\infty} |a_n x^n|$ converges. Applying the n-th root test,

$$\overline{\lim}\ \sqrt[n]{|a_n x^n|} = \overline{\lim}\ \sqrt[n]{|a_n|}|x| \le 1$$

so that $|x| \le 1/\overline{\lim}\ \sqrt[n]{|a_n|}$.

If $\rho = \overline{\lim}\ \sqrt[n]{|a_n|} = 0$, then $\overline{\lim}\ \sqrt[n]{|a_n x^n|} = 0$, which implies that the series $\sum_{n=0}^{\infty} a_n x^n$ converges for all x and hence that $R = \infty$. If $\rho \ne 0$ and $|x| > 1/\rho$, then $|x|\rho > 1$ so that $\sqrt[n]{|a_n|}|x| \ge 1$ for infinitely many n which implies that $|a_n x^n| \ge 1$ for infinitely many n and hence the divergence of $\sum_{n=0}^{\infty} a_n x^n$. **QED**.

Example 1. Since $\sqrt[n]{n!} \to \infty$, the series $\sum_{n=0}^{\infty} n! x^n$ has $R = 0$ while $\sum_{n=0}^{\infty} x^n/n!$ has $R = \infty$. The series $\sum_{n=0}^{\infty} x^n$ has $R = 1$.

If $a_n \ne 0$ for $n \ge N$ and $L = \lim |a_{n+1}/a_n|$ exits (possibly $= \infty$), then the radius of convergence of $\sum_{n=0}^{\infty} a_n$ is $1/L$.

Example 2. The series $\sum_{n=0}^{\infty} 2^n x^{2n}$ is a power series $\sum_{n=1}^{\infty} a_n x^n$ with $a_{2n+1} = 0$, $a_{2n} = 2^n$. Since $\overline{\lim}\ \sqrt[n]{|a_n|} = \overline{\lim}\ \sqrt[2n]{2^n} = \sqrt{2}$, we have $R = 1/\sqrt{2}$ while the ratios a_{n+1}/a_n cannot be computed since $a_n = 0$ for n odd.

However, both the ratio and root test for the series $\sum_{n=0}^{\infty} a_n$ with $a_n = 2^n|x|^{2n}$ apply to give convergence for $2|x|^2 < 1$ and divergence for $2|x|^2 > 1$. This gives $R = 1/\sqrt{2}$.

Theorem. If I is the interval of convergence of $\sum_{n=0}^{\infty} a_n x^n$, then the series converges uniformly on any closed and bounded subset K of I.

Proof. If R is the radius of convergence and $K \subset (-R, R)$ then there is an $r < R$ such that $x \in K$ implies that $|x| \le r$. Then $\sum |a_n x^n| \le \sum |a_n r^n|$ and the result follows by the Weierstrass M-test. If $R \in K$, we have to show that the convergence is uniform on $[0, R]$. Similarly, if $-R \in K$, we have to show the convergence is uniform. The result would then follow since K could be expressed as the union of two sets on each of which the convergence is uniform.

After a change of variable of the form $s = \pm x/R$ we may assume $R = 1$. We are then reduced to proving that the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$ if $\sum_{n=0}^{\infty} a_n$ converges. Let $\epsilon > 0$ be given and choose N so that $|\sum_{n=N}^{\infty} a_n| < \epsilon$. Then by Abel's test, with $b_n = x^n$, we have $|\sum_{n=N}^{\infty} a_n x^n| \le \epsilon$. **QED**

Corollary. If I is the interval of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in I$, then f is continuous on I and, we have, for $a, b \in I$,

$$\int_a^b f(x)\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(b^{n+1} - a^{n+1}).$$

In particular, $\int_0^x f(t)\, dt = \sum_{n=1}^{\infty} a_{n-1} x^n/n = a_0 x + a_1 x^2/2 + a_2 x^3/3 + \cdots$.

Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$ the radius of convergence R of the series $\sum_{n=1}^{\infty} a_{n-1}x^n/n$ is the same as that of $\sum_{n=0}^{\infty} a_n x^n$. The same is true for the series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

obtained by term by term differentiation from the series $\sum_{n=0}^{\infty} a_n x^n$. The function

$$g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

is continuous for $|x| < R$ and $\int_0^x g(t)\,dt = f(x)$ for $|x| < R$. Hence $f'(x) = g(x)$. We thus obtain the following result.

Theorem. If R is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, we have, for $|x| < R$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \implies f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Corollary. If $R > 0$, we have $f^{(n)}(0) = n! a_n$. In particular, if $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ on some interval $(-r, r)$ with $r > 0$, then $a_n = b_n$ for all n.

Example 3. The series $\sum_{n=0}^{\infty} (-1)^n x^n$ has $(-1, 1)$ for interval of convergence and

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \implies \log(1+x) = \int_0^x \frac{dt}{1+t} = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

if $|x| < 1$. Since the power series on the right has interval of convergence $(-1, 1]$ and is a continuous function of x there, the above series expansion for $\log(1+x)$ is valid on $(-1, 1]$. In particular,

$$\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Example 4. The series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ has $(-1, 1)$ for interval of convergence and

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \implies \tan^{-1}(x) = \int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

if $|x| < 1$. Since the power series on the right has interval of convergence $(-1, 1]$ and is a continuous function of x there, the above series expansion for $\tan^{-1}(x)$ is valid on $(-1, 1]$. In particular,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

If $f(x)$ is infinitely differentiable on some open interval I containing c, the series

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the Taylor series of f at c. This series will converge to $f(x)$ if and only if the remainder term $R_n(x)$ in Taylor's formula converges to zero. If this is the case on an open interval I, we say that the Taylor series represents f on I.

Example 5. The function f defined by $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$ is infinitely differentiable on $\mathbb{R}$ and $f^{(n)}(0) = 0$ for all n. Thus the Taylor series for $f(x)$ at 0 converges for all x but it does not represent f on any open non-empty interval.

Example 6. If $f(x) = e^x$, then $R_n(x) = e^{\theta x}/n!$ with $0 < \theta < x$. Hence $0 < R_n(x) < e^x/n! \to 0$ which implies that the Taylor series for e^x at $x = 0$, namely $\sum_{n=0}^{\infty} x^n/n!$, represents e^x

Exercise. If $f(x) = \sum_{n\geq 0} a_n x^n$ has radius of convergence 1 with $a_n \geq 0$ for all n and $\sum_{n\geq 0} a_n$ divergent, show that $f(x) \to \infty$ as $x \to 1-$. **Hint:** Show that $f(x)/(1-x) \geq (\sum_{n\leq N} a_n)(\sum_{n\geq N} x^n)$.