MATH 255: Lecture 19

Positive Series:The Integral Test, the Kummer-Jensen Tests

A powerful test for the convergence or divergence of a positive series is the integral test. If f is a positive, decreasing function on $[1,\infty)$ and $a_n = f(n)$, we have

$$\sum_{k=2}^{n} a_k = \int_1^n f(x)\,d[x] = \int_1^n f(x)\,dx - \int_1^n f(x)\,d((x)) = \int_1^n f(x)\,dx + \int_1^n ((x))\,df(x),$$

where $((x)) = x - [x]$. Since $-f$ is increasing and $0 \le ((x)) \le 1$, we have

$$0 \le -\int_1^n ((x))\,df(x) = \int_1^n ((x))\,d(-f(x)) \le f(1) - f(n).$$

Thus

$$0 \le \int_1^n f(x)\,dx - \sum_{k=2}^{n} a_k \le a_1 - a_n \implies \sum_{k=2}^{n} a_k \le \int_1^n f(x)\,dx \le \sum_{k=1}^{n-1} a_k$$

so that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} \int_1^n f(x)\,dx$ exists. More generally, if f is integrable on $[a,b]$ for all $b \ge a$ and $\lim_{b\to\infty} \int_a^b f(x)\,dx$ exists, we define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

Such an integral with an infinite upper limit is an example of a convergent improper integral. The integral is said to be divergent if the above limit does not exist.

Theorem (Integral Test). If f is positive and decreasing on $[1,\infty]$ then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \iff \int_1^\infty f(x)\,dx \quad \text{converges},$$

in which case

$$r_n = \sum_{k=n+1}^{\infty} a_k \le \int_n^\infty f(x)\,dx.$$

Example 1. If we apply the integral test to the p-series, we have $f(x) = 1/x^p$. Since

$$\int_1^n \frac{dx}{x^p} = \begin{cases} \log n & \text{if } p = 1, \\ \frac{n^{1-p}}{1-p} - \frac{1}{1-p} & \text{if } p \ne 1, \end{cases}$$

we see that $\sum 1/n^p$ converges if and only if $p > 1$ and that in this case $r_n \le 1/(p-1)n^{p-1}$.

Both the ratio and root tests amount to a comparison with a geometric series but are inconclusive when the ratio or root approaches 1 from below. Using telescoping series one can obtain sharper tests. A series of the form $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ is called a telescoping series since

$$\sum_{k=1}^{n}(a_k - a_{k+1}) = a_1 - a_{n+1}.$$

Such a series converges if and only if $L = \lim a_n$ exists in which case the sum of the series is $a_1 - L$.

Theorem (Kummer's Test). If (c_n) is any positive series, the strictly positive series $\sum a_n$ will converge if

$$K_n = c_n - c_{n+1}\frac{a_{n+1}}{a_n} \ge h > 0 \text{ for } n \ge N.$$

Proof. Since $0 < ha_n \le b_n = c_n a_n - c_{n+1}a_{n+1}$ for $n \ge N$, the positive sequence $(c_n a_n)$ is decreasing for $n \ge N$ and so is convergent. Thus the telescoping series $\sum b_n$ is convergent and $\sum a_n << \sum b_n$. **QED**

Theorem (Jensen's Test). If $\sum 1/c_n$ is a positive divergent series, the strictly positive series $\sum a_n$ will diverge if

$$K_n = c_n - c_{n+1}\frac{a_{n+1}}{a_n} \le 0 \text{ for } n \ge N.$$

Proof. For $n \ge N$ we have $c_n a_n \ge c_N a_N$ and so $a_n \ge C/c_n$ with $C = c_N a_N$. **QED**

The limit form of these tests can be combined into the following theorem.

Theorem. Let (a_n), (c_n) be strictly positive series and let $K_n = c_n - c_{n+1}a_{n+1}/a_n$. Then

$$\text{(a)}\quad \underline{\lim}\, K_n > 0 \implies \sum a_n \text{ converges },\qquad \text{(b)}\quad \overline{\lim}\, K_n < 0 \implies \sum a_n \text{ diverges with } \sum \frac{1}{c_n}.$$

The proof is left to the reader. This theorem now yields various test by choosing different sequences (c_n).

1. If $c_n = 1$, then $K_n = 1 - a_{n+1}/a_n$ and we get d'Alembert's test.

2. If $c_n = n-1$, then $K_n = n(1 - a_{n+1}/a_n) - 1$. Hence, if we put

   $$R_n = K_n + 1 = n(1 - a_{n+1}/a_n), \text{ we get}$$

   Raabe's Test: $\underline{\lim}\, R_n > 1 \implies \sum a_n$ converges, $\overline{\lim}\, R_n < 1 \implies \sum a_n$ diverges.

3. If $c_n = (n-1)\log(n-1)$, then $K_n = (n-1)\log\frac{n-1}{n} + B_n$, where

   $$B_n = \left(n\left(1 - \frac{a_{n+1}}{a_n}\right) - 1\right)\log n = (R_n - 1)\log n.$$

   Since $(n-1)\log\frac{n-1}{n} \to -1$, we get

   Bertrand's Test. $\underline{\lim}\, B_n > 1 \implies \sum a_n$ converges, $\overline{\lim}\, B_n < 1 \implies \sum a_n$ diverges.

Example 2. For the series $\displaystyle\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} = 1 + \alpha + \frac{\alpha(\alpha+1)}{2} + \cdots$ we have, for $\alpha \ne 0$, $\displaystyle\frac{a_{n+1}}{a_n} = \frac{\alpha+n}{n+1}$ and $\displaystyle R_n = n\left(1 - \frac{\alpha+n}{n+1}\right) = \frac{n(1-\alpha)}{n+1} \to 1-\alpha$. Since $a_{n+1}/a_n > 0$ for $n > -\alpha$, the terms have the same sign for $n \ge N$ and we can apply Raabe's Test to get convergence if $1-\alpha > 1$ ($\alpha < 0$) and divergence if $1 - \alpha < 1$ ($\alpha > 0$). If $\alpha = 1$, then $a_n = 1$ and we have divergence.

Example 3. In the series

$$\sum_{n=1}^{\infty} \frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)}\cdot\frac{1}{2n+2},$$

$$\frac{a_{n+1}}{a_n} = \frac{2n+2}{2n+1}\cdot\frac{2n+2}{2n+4} = \frac{4n^2+8n+4}{4n^2+10n+4},$$

$$R_n = n\left(1 - \frac{a_{n+1}}{a_n}\right) = \frac{2n^2}{4n^2+10n+4} \to \frac{1}{2}.$$

and the series diverges.

More generally, if $\displaystyle\frac{a_{n+1}}{a_n} = \frac{n^k + bn^{k-1} + \cdots}{n^k + cn^{k-1} + \cdots}$, the ratio test fails but

$$R_n = n\left(1 - \frac{a_{n+1}}{a_n}\right) = \frac{(c-b)n^k + \cdots}{n^k + \cdots} \to c - b.$$

By Raabe's test, the series $\sum a_n$ converges if $c - b > 1$ and diverges if $c - b < 1$. When $c - b = 1$,

$$B_n = (R_n - 1)\log n = \frac{\log n}{n} \cdot \frac{rn^{k-1} + \cdots}{n^{k-1} + \cdots} \to 0,$$

and the series diverges by Bertrand's test. More generally, we have

Theorem (Gauss' Test). If $R_n = h + O(1/n^p)$ with $p > 0$, then $\sum a_n$ converges if $h > 1$ and diverges if $h < 1$.

The proof is left to the reader. If (a_n), (b_n) are positive sequences with $b_n > 0$, then

$$\begin{aligned} a_n = O(b_n) &\iff (\exists M, N)(\forall n \geq N)\ a_n \leq M b_n \\ a_n = o(b_n) &\iff \lim_{n\to\infty} \frac{a_n}{b_n} = 0. \end{aligned}$$

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