MATH 255: Lecture 18

Positive Series: Comparison, Ratio and n-th Root Tests

A series $\sum a_n$ is said to be positive if $a_n \geq 0$ for all n. Since the series $\sum |a_n|$ is a positive series any test for the convergence of a positive series will result in a test for absolute convergence. Since the convergence or divergence of a series is unaffected by omitting a finite number of terms, any test for positive series can be applied to a series having only a finite number of negative terms. A positive series converges if and only if the partial sums are bounded. This fact is the basis for the comparison test.

Theorem (Comparison Test). Suppose $0 \leq a_n \leq b_n$ for all n. Then

(a) The convergence of $\sum b_n$ implies the convergence of $\sum a_n$.

(b) The divergence of $\sum a_n$ implies the divergence of $\sum b_n$.

If $\sum a_n$ and $\sum b_n$ are positive series we say that the first series is dominated by the second or that the second dominates the first if there is a $C > 0$ and an N such that $a_n \leq Cb_n$ for $n \geq N$. We denote this by

$$\sum a_n << \sum b_n.$$

In this case $\sum b_n$ converges $\Rightarrow \sum a_n$ converges and $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges.

Example 1. Since $\frac{1}{2^n-1} \leq \frac{2^n}{2^n-1}\frac{1}{2^n} \leq 2\frac{1}{2^n}$ for $n \geq 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n-1} << \sum_{n=1}^{\infty} \frac{1}{2^n}$$

and so is convergent.

Theorem (Cauchy Condensation Test). If (a_n) is a positive decreasing sequence, the series

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} 2^n a_{2^n}$$

both converge or both diverge.

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ and $S_n = \sum_{k=0}^{n-1} 2^k a_{2^k}$ be respectively the n-th partial sums of the two series. Then

$$s_n \leq s_{2^n-1} \leq S_n \text{ since } a_k \leq a_{2^{m-1}} \text{ for } 2^{m-1} \leq k < 2^m.$$

Also $S_n < 2s_{2^{n-1}}$ since

$$2^k a_{2^k} \leq 2 \sum_{m=2^{k-1}+1}^{2^k} a_m.$$

QED

Example 2 (The p-series). Applying the Cauchy Condensation Test to the p-series, we get the series

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=0}^{\infty} \frac{1}{(2^p)^n},$$

which is a geometric series with ratio $r = 1/2^p$. Hence the p-series converges if and only if $p > 1$.

Example 3. Applying the Cauchy Condensation Test to $\sum \frac{1}{n(\log n)^c}$, we get

$$\sum_{n=1}^{\infty} \frac{2^n}{n^c(\log 2)^c} = \frac{1}{(\log 2)^c} \sum_{n=1}^{\infty} \frac{1}{n^c}$$

which converges $\Leftrightarrow c > 1$.

Theorem (Limit Form of Comparison Test). If $\sum a_n$ and $\sum b_n$ are strictly positive series and the ratio

$$\frac{a_n}{b_n} \to r > 0 \text{ as } n \to \infty,$$

the series both converge or both diverge. If $a_n/b_n \to 0$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n/b_n \to \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof. Since $r > 0$ we can choose ϵ so that $0 < \epsilon < r$. Then N can be found so that, for $n \geq N$,

$$r - \epsilon < \frac{a_n}{b_n} < r + \epsilon$$

which implies $(r - \epsilon)b_n < a_n < (r + \epsilon)b_n$ for $n \geq N$.

When $a_n/b_n \to 0$, we have $a_n/b_n < \epsilon$ for $n \geq N$ and so $a_n < \epsilon b_n$ for $n \geq N$.

When $a_n/b_n \to \infty$, we have $a_n/b_n > M > 0$ for $n \geq N$ and so $a_n > Mb_n$ for $n \geq N$. **QED**

Corollary (Polynomial Test). If $P(n)$ and $Q(n)$ are monic polynomials in n of degree p and q respectively, then the infinite series

$$\sum \frac{P(n)}{Q(n)} = \sum \frac{n^p + \cdots}{n^q + \cdots}$$

converges $\Leftrightarrow q > p + 1$.

Proof. Applying the limit form of the comparison test with $a_n = P(n)/Q(n)$ and $b_n = n^p/n^q = 1/n^{q-p}$, we get $a_n/b_n \to 1 \neq 0$. Note that $a_n > 0$ for $n \geq N$. **QED**

Theorem (Cauchy's Root Test). Given The positive series $\sum a_n$, if

(a) $\sqrt[n]{a_n} \leq r < 1$ for $n \geq N$, the series converges;

(b) $\sqrt[n]{a_n} \geq 1$ for infinitely many n, the series diverges.

Proof. In case (a), we have $a_n \leq r^n$ for $n \geq N$ with $r < 1$. In case (b), we have $a_n \geq 1$ for infinitely many n which shows that $a_n \nrightarrow 0$. **QED**

If $a_n = 1/n^p$ and $p > 0$, then $\sqrt[n]{a_n} < 1$ for all n and converges to 1 from below so that the root test gives no information.

If a_n is a bounded sequence of real numbers, the sequence $b_n = \sup_{k \geq n} a_k$ is a decreasing sequence which is bounded below and so converges. The limit is called the lim sup or upper limit of the sequence (a_n). It is denoted by $\limsup_{n\to\infty} a_n$ or $\overline{\lim}\, a_n$. Since

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k = \inf_n \sup_{k\geq n} a_k,$$

The upper limit of the sequence (a_n) is the smallest number r such that

$$(\forall \epsilon > 0)(\exists N) \quad a_n < r + \epsilon \text{ for } n \geq N$$

Similarly, since $c_n = \inf_{k\geq n} a_k$ is an increasing sequence, one can define the lower limit or lim inf of the positive sequence (a_n) by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k = \sup_n \inf_{k\geq n} a_k,$$

The lower limit of the sequence (a_n) is the largest number r such that

$$(\forall \epsilon > 0)(\exists N) \quad a_n > r - \epsilon \text{ for } n \geq N$$

We have $\overline{\lim}\, a_n = \underline{\lim}\, a_n$ if and only lim a_n exists.

Exercise 1. A number r is called a limit point of the sequence (a_n) if every interval I containing x has the property that there are infinitely many n with $a_n \in I$. Show that the set of limit points of the sequence (a_n) is closed and is bounded if (a_n) is bounded, in which case the upper and lower limits of the sequence are respectively the largest and smallest limit points of the sequence.

If (a_n) is not bounded above, we define $\overline{\lim}\, a_n = +\infty$ and define $\underline{\lim}\, a_n = -\infty$ if the sequence is not bounded below. By convention, we let $-\infty < a < +\infty$ for any $a \in \mathbb{R}$.

Exercise 2. If (a_n), (b_n) are positive sequences and $b_n > 0$, show that

(a) if $\overline{\lim}\, \frac{a_n}{b_n} < \infty$, then $\sum b_n$ converges $\implies \sum a_n$ converges;

(b) if $\underline{\lim}\, \frac{a_n}{b_n} > 0$, then $\sum b_n$ diverges $\implies \sum a_n$ diverges.

Deduce that if $0 < \underline{\lim}\, \frac{a_n}{b_n} \leq \overline{\lim}\, \frac{a_n}{b_n} < \infty$, then $\sum a_n$ converges $\iff \sum b_n$ converges.

Theorem (Limit form of Root Test). Let (a_n) be a bounded positive sequence and let $r = \overline{\lim}\, \sqrt[n]{a_n}$. Then $\sum a_n$ converges if $r < 1$ and diverges if $r > 1$.

The proof is left as an exercise.

Theorem (d'Alembert's Ratio Test.) Let $\sum a_n$ be a strictly positive series.

(a) If $a_{n+1}/a_n \leq r < 1$ when $n \geq N$, the series diverges;

(b) If $a_{n+1}/a_n \geq 1$ when $n \geq N$, the series diverges.

Proof. In case (a), we have $a_{N+n} \leq a_N r^n$ for $n \geq 0$. In case (b), we have $a_{n+1} \geq a_n$ for $n \geq N$ so that $a_n \nrightarrow 0$. **QED**

Theorem (Limit form of Ratio Test). If $a_n > 0$ for all n, the series $\sum a_n$ converges if $\overline{\lim}\, a_{n+1}/a_n < 1$ and diverges if $\underline{\lim}\, a_{n+1}/a_n > 1$.

The proof is left as an exercise.

Example 1. Consider the series

$$1 + b + bc + b^2c + b^2c^2 + \cdots + b^nc^{n-1} + b^nc^n + \cdots,$$

where $0 < b < c$. Then $a_{n+1}/a_n = b$ when n is odd and $= c$ when n is even. Then $\underline{\lim}\, a_{n+1}/a_n = b$ and $\overline{\lim}\, a_{n+1}/a_n = c$ and so the series converges if $c < 1$ and diverges if $b > 1$ but the ratio test gives no information if $b \leq 1 \leq c$. However $\sqrt[n]{a_n} \to \sqrt{bc}$ so that the series converges if $bc < 1$ and diverges if $bc > 1$. When $bc = 1$ the series becomes $1 + b + 1 + b + \cdots$ which diverges.

Exercise 3. If $a_n > 0$, show that

$$\text{(a)} \quad \overline{\lim}\, \sqrt[n]{a_n} \leq \overline{\lim}\, \frac{a_{n+1}}{a_n}, \qquad \text{(b)} \quad \underline{\lim}\, \frac{a_{n+1}}{a_n} \leq \underline{\lim}\, \sqrt[n]{a_n}.$$

Deduce that $\lim_{n\to\infty} a_{n+1}/a_n = r \Rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = r$

Exercise 4. If $a_n = n!/n^n$, show that $a_{n+1}/a_n \to 1/e$. Use this to show that $\sqrt[n]{n!}/n \to 1/e$.

The ratio test is less powerful than the root test but is easier to use in general. When successful in proving convergence a series $\sum a_n$ they yield estimates for a_n which can be used in the estimation of the sum of the series. For the ratio test and root tests we have respectively

$$a_n \leq \frac{a_N}{r^N} r^n \quad \text{and} \quad a_n \leq r^n$$

for $n \geq N$ and some $0 < r < 1$. If $s = \sum_{n=1}^{\infty} a_n$, then $r_n = s - s_n = \sum_{k=n+1}^{\infty} a_k$ so that in the case of the ratio test $r_n \leq a_n r/(1-r)$ for $n \geq N$ and $r_n \leq r^{n+1}/(1-r)$ for $n \geq N$ in the case of the root test.

L'Hospital's rule is useful in the computation of the limits involved in the use of the various convergence tests but some simple principles can often simplify the work. If (a_n) and (b_n) sequences and $b_n \neq 0$ for all $n \geq N$, we say that a_n is asymptotic to b_n as $n \to \infty$ if the ratio a_n/b_n converges to 1 as $n \to \infty$. We denote this by $a_n \sim b_n$. For example, $\sqrt[n]{n!} \sim n/e$. If $a_n \sim b_n$ then

$$\lim_{n\to\infty} a_nc_n = \lim_{n\to\infty} \frac{a_n}{b_n} b_nc_n = \lim_{n\to\infty} b_nc_n$$

so that, in computing a limit, a factor can be replaced by something asymptotic to it. If $a_n \sim b_n$ then $1/a_n \sim 1/b_n$ and, if $c_n \sim d_n$, then $a_n c_n \sim b_n c_n$.

For example, if $P(n) = an^p + \cdots$, $Q(n) = bn^q + \cdots$ are polynomials of degree p and q respectively, then $P(n) \sim an^p$, $Q(n) \sim bn^q$ and $P(n)/Q(n) \sim an^{p-q}/b$. Hence,

$$\lim_{n\to} \frac{n^3 - 2n^2 + 3n + 1}{n^2 - 2n + 5} \sin(1/n) = \lim_{n\to\infty} n \sin(1/n) = 1.$$

The extended real number system. The real numbers together with two new elements, denoted by $-\infty$ and ∞ is called the extended real number system. We extend to it the order relation on the reals by $-\infty < r < \infty$ for all real r. The operations on the reals are partially extended by

(a) $r + \infty = \infty + r = \infty$, $r + (-\infty) = -\infty + r = -\infty$, $\dfrac{r}{\pm\infty} = 0$;

(b) If $r > 0$, then $r \cdot \infty = \infty \cdot r = \infty$ $r \cdot (-\infty) = -\infty \cdot r = -\infty$, $\dfrac{r}{0} = \infty$;

(c) If $r < 0$, then $r \cdot \infty = \infty \cdot r = -\infty$ $r \cdot (-\infty) = -\infty \cdot r = \infty$, $\dfrac{r}{0} = -\infty$;

(d) $\infty \cdot \infty = \infty$, $-\infty \cdot \infty = \infty \cdot (-\infty) = -\infty$, $-\infty \cdot (-\infty) = \infty$.

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