MATH 255: Lecture 14

The Elementary Transcendental Functions: The Exponential and Log Functions

The elementary transcendental functions are the functions that can be obtained from the exponential function

a^x, the circular functions $\sin x$ and $\cos x$ and their associated inverse functions by means of rational operations and composition of functions. In this lecture we will treat the exponential function.

Let a be a real number > 1. One defines a^x for a natural number $x = n \in \mathbb{N}$ by induction:

$$a^0 = 1, \quad a^{n+1} = a^n a$$

and extends this to all integers $x \in \mathbb{Z}$ by defining $a^{-n} = 1/a^n = (1/a)^n$. For integral x, y we have

$$a^{x+y} = a^x a^y.$$

In the case $x = 1/n$ with n a natural number, one defines $a^{\frac{1}{n}}$ to be the unique solution $b > 1$ of the equation $b^n = a$. The existence of b follows from the fact that the function $f(x) = x^n$ is an increasing continuous function which maps $\mathbb{R}$ onto $\mathbb{R}$. For rational $x = m/n \in \mathbb{Q}$, one defines

$$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m.$$

It is left to the reader to show the identity $a^{x+y} = a^x a^y$ is preserved. For an arbitrary real number x one defines a^x by

$$a^x = \sup_{r \in \mathbb{Q},\ r<x} a^r.$$

It follows that $a^{x+y} = a^x a^y$ for arbitrary $x, y \in \mathbb{R}$ and that $a^{xy} = (a^x)^y$. In particular,

$$(\frac{1}{a})^x = a^{-x} = 1/a^x.$$

To prove the continuity of a^x, it therefore suffices to prove its right continuity at $x = 0$. This follows from the fact that

$$\lim_{n \to \infty} a^{\frac{1}{n}} = 1.$$

To see this, let $a^{1/n} = 1 + h_n$. Then $h_n > 0$ and $1 + nh_n < a$ which shows that $h_n < (a-1)/n$ which tends to zero as $n \to \infty$.

The function a^x is strictly increasing and maps the positive reals $\mathbb{R}_{>0}$ onto the reals $\mathbb{R}$. It therefore has an inverse, $\log_a x$, the logarithm function to the base a. We have

$$a^{\log_a x} = x, \quad \log_a(a^x) = x.$$

To prove that that the functions a^x and $\log_a^x$ are differentiable, we have to show that the limits

$$\lim_{h \to 0} \frac{a^{x+h} - a^x}{n} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} \text{ and}$$
$$\lim_{h \to 0} \frac{\log_a(x+h) - \log_a x}{h} = \frac{1}{x} \lim_{h \to 0} \log_a(1 + \frac{h}{x})^{x/h}$$

exist.

Theorem. We have $\lim_{h \to \infty}(1 + 1/h)^h = e$ where $e = \sum_{n=0}^{\infty} 1/n!$.

Proof. Let $x_n = (1 + 1/n)^n$, let $y_n = \sum_{k=0}^{n} 1/k!$ and, for $m < n$, let

$$x_{n,m} = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{m-1}{n}) \le x_n.$$

Then $x_{n+1} > x_{n+1,n} > x_n$ and $x_n < y_n < 1 + \sum_{k=0}^{n-1} 1/2^k < 3$. thus (x_n) and (y_n) are bounded increasing sequences and hence convergent. Since $x_{m,n} \le x_n$ and $\lim_{n\to\infty} x_{m,n} = y_m$, we see that $\lim x_n = \lim y_n$=e. If $x > 0$ and $n = [x]$, we have

$$\left(1 + \frac{1}{n+1}\right)^n < (1 + \frac{1}{x})^x < (1 + \frac{1}{n})^{n+1},$$

which shows that $\lim_{x\to\infty}(1 + 1/x)^x = e$ since

$$\lim_{n\to\infty}\left(1 + \frac{1}{n+1}\right)^n = \lim_{n\to\infty}(1 + \frac{1}{n})^{n+1} = e.$$

This shows that $\lim_{h\to 0+}(1 + h)^{1/h} = e$. Since

$$\lim_{x\to -\infty}(1 + \frac{1}{x})^x = \lim_{x\to\infty}(1 - \frac{1}{x+1})^{-(x+1)} = \lim_{x\to\infty}(1 + \frac{1}{x})(1 + \frac{1}{x})^x = e,$$

we see that $\lim_{h\to 0-}(1 + h)^{1/h} = e$ and hence that $\lim_{h\to 0}(1 + h)^{1/h} = e$ **QED**

Corollary. We have

$$\frac{d}{dx}\log_a x = \frac{\log_a e}{x}, \quad \frac{d}{dx}a^x = a^x \log_e a.$$

In particular, $\frac{d}{dx}\log_e x = \frac{1}{x}, \quad \frac{d}{dx}e^x = e^x$.

Since the function e^x is the inverse function to $\log_e x$, we get $\frac{d}{dx}e^x = e^x$. To prove the statement about a^x, we then use the fact that $a^x = e^{x \log_e a}$. Note that this yields $\lim_{h\to 0}(a^h - 1)/h = \log_e a$.

Note. It is customary to denote $\log_e x$ by $\log x$ or $\ln x$.

Corollary. The function $y = e^x$ is the unique solution to the initial value problem $y' = y$, $y(0) = 1$. In particular, $e^x = \exp(x) = \sum_{n=0}^{\infty} x^n/n!$.

Corollary. The function $y = e^{ax}$ is the unique solution to the initial value problem $y' = ay$, $y(0) = 1$. In particular, $e^{ax} = \exp(ax) = \sum_{n=0}^{\infty} a^n x^n/n!$.