MATH 255: Lecture 13

Sequences of Functions: Uniform Convergence and Differentiation

If $f_n(x) = x^n/n$, the sequence (f_n) converges uniformly to the function $f = 0$ on $[0, 1]$. However, $f'_n(x) = x^{n-1}$ so that the sequence of derivatives (f'_n) converges pointwise to the function g, where $g(x) = 0$ if $x \neq 1$ and $g(1) = 1$. Since $f' \neq g$, this shows that one cannot, in general, interchange limits and derivatives. Nevertheless, one has the following result:

Theorem. Let (f_n) be a sequence of differentiable functions on a bounded interval I such that $(f_n(x_0))$ converges for some point $x_0 \in I$. If the sequence (f'_n) converges uniformly on I to a function g, then (f_n) converges uniformly on I to a function f which is differentiable on I and $f' = g$.

Proof. Let $a < b$ be the endpoints of I and let $\epsilon > 0$ be given. Choose N_1 so that, for $m, n \geq N_1$

$$|f'_m(x) - f'_n(x)| < \min(\frac{\epsilon}{3}, \frac{\epsilon}{3(b-a)}) \quad \text{and} \quad |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{3}.$$

For any $x \in I$, apply the Mean Value Theorem for Derivatives to the interval with endpoints x_0, x to get

$$f_m(x) - f_n(x) = f_m(x_0) - f_n(x_0) + (x - x_0)(f'_m(y) - f'_n(y))$$

for some y between x_0 and x. Hence

$$|f_m(x) - f_n(x)| \leq |f_m(x_0) - f_n(x_0)| + (b-a))|f'_m(y) - f'_n(y)| < \frac{2\epsilon}{3} < \epsilon.$$

This shows that (f_n) is uniformly convergent on I. If $f = \lim f_n$, the f is continuous on I since each f_n is continuous on I.

To show that f is differentiable at a point $c \in I$, we apply the Mean Value Theorem for Derivatives to $f_m - f_n$ on the interval with endpoints $x, c \in I$ with $x \neq c$. We get

$$(f_m(x) - f_n(x)) - (f_m(c) - f_n(c)) = (x - c)(f'_m(z) - f'_n(z))$$

with z between x and c. Dividing by $x - c$ and taking absolute values, we get

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq |f'_m(z) - f'_n(z)| < \frac{\epsilon}{3}.$$

Passing to the limit with respect to m, we get

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \frac{\epsilon}{3}.$$

Now choose N_2 so that $|f'_n(c) - g(c)| < \frac{\epsilon}{3}$ for $n \geq N_2$. Let $N = \max(N_1, N_2)$ and choose $\delta > 0$ so that

$$0 < |x - c| < \delta \implies \left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\epsilon}{3}.$$

Then, combining these inequalities, we get

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that f' exists on I and equals g.

Corollary. If $\sum_{n=1}^{\infty} f_n$ is a series of differentiable functions on a bounded interval I such that

(a) $\displaystyle\sum_{n=1}^{\infty} f_n(x_0)$ converges for some point $x_0 \in I$,

(b) $g = \displaystyle\sum_{n=1}^{\infty} f_n'$ is uniformly convergent on J,

then $f = \displaystyle\sum_{n=1}^{\infty} f_n$ is uniformly convergent and $f' = g$.

While the uniform convergence of a sequence of continuous functions is sufficient for the limit function to be continuous, it is not necessary. For example, if

$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le 1/n, \\ -n(x - 2/n) & \text{if } 1/n \le x \le 2/n, \\ 0 & \text{if } 2/n \le x \le 2, \end{cases}$$

the sequence (f_n) converges pointwise to the zero function on $[0, 2]$ but the convergence is not uniform as $f_n(1/n) = 1$. However, for monotone sequences of continuous functions on a closed interval, uniform convergence is necessary if the pointwise limit exists and is continuous.

Theorem. (Dini) Let (f_n) be a monotone sequence of continuous functions on a closed interval $[a, b]$. If (f_n) converges pointwise to a continuous function f, the the convergence is uniform.

Proof. Possibly replacing (f_n) by $(-f_n)$, we can assume that $f_{n+1}(x) \le f_n(x)$. Also, after replacing f_n by $f_n - f$, we can assume $f = 0$. Let $\epsilon > 0$ be given. For each $t \in [a, b]$, there is an $N(t) > 0$ such that $f_{N(t)}(t) < \epsilon$. By continuity, there is a $\delta(t) > 0$ such that $f_{N(t)}(x) < \epsilon$ if $|x - t| < \delta(t)$. Now let (P, t) be a δ-fine tagged partition for the gauge δ constructed above and let $N = \min(N(t_1), \ldots, t_n)$, where $t = (t_1, \ldots, t_n)$. Then, for any $x \in [a, b]$, we have $f_N(x) < \epsilon$ since $|x - t_k| < \delta(t_k)$ for some k. Hence $f_n(x) < \epsilon$ for $n \ge N$ and all $x \in [a, b]$. **QED**

Note that the the example $f_n(x) = x^n$ on $(0, 1)$ shows that the result is false on an interval which is not closed.

(Last updated 2:00 pm April 14, 2003)