MATH 255: Lecture 11

Sequences of Functions: Pointwise and Uniform Convergence

The study of functions defined by means of differential equations is a central problem in mathematics. Consider for example, the differential equation

$$\frac{dy}{dx} = y.$$

A solution to this equation is a function $y = f(x)$ with $f' = f$. It we integrate this equation on $[0, x]$, we get

$$f(x) = f(0) + \int_0^x f(t)\, dt$$

which is an integral equation which has the same solutions as the original differential equation. We now describe an iterative process for constructing a solution of this integral equation.

Let $f_0, f_1, \ldots, f_n, \ldots$ be the sequence of functions defined by

$$f_0(x) = C, \qquad f_{n+1}(x) = C + \int_0^x f_n(t)\, dt.$$

Then $f_1(x) = C(1 + x)$, $f_2(x) = C(1 + x + x^2/2)$, $f_n = C(1 + x + x^2/2 + x^3/6 + \cdots + x^n/n!)$. Suppose that we could show that the sequence $(f_n(x))_{n\geq 0}$ converged for each x to $f(x)$. In this case we would say that the sequence of functions (f_n) converged pointwise to the function f. Passing to the limit in the above integral equation we would have

$$f(x) = C + \lim_{n\to\infty} \int_0^x f_n(t)\, dt.$$

If we could interchange the limit and the integral, we would have a solution to our integral equation since $f(0) = C$. The justification of this last step uses the fact that the sequence (f_n) converges "uniformly" to f. We thus obtain that

$$f(x) = C(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots) = C\sum_{n=0}^{\infty} x^n/n!$$

is a solution to our differential equation. We also see that $f(0) = 0$ implies that $f = 0$. This shows that any two solutions f, g with $f(0) = g(0)$ must be equal. Indeed, $h = f - g$ is then a solution and $h(0) = 0$. Thus the initial value problem $y' = y$, $y(0) = C$ has the unique solution $y = C(\sum_{n=0}^{\infty} x^n/n!)$.

Definition. A sequence (f_n) of real-valued functions f_n defined on a set A is said to converge pointwise on A if for each $x \in A$, the sequence $(f_n(x))$ converges. If (f_n) converges pointwise on A and we set, for each $x \in A$,

$$f(x) = \lim_{n\to\infty} f_n(x),$$

we obtain a function f on A. When such a function exists, we say that (f_n) converges to f and write

$$f = \lim f_n \text{ on } A \quad \text{or} \quad f_n \to f \text{ on } A.$$

Example 1. Let $f_n(x) = x^n$. Then $f_n(x)$ converges if and only if $x \in A = (-1, 1]$ with limit f where $f(1) = 1$ and $f(x) = 0$ for $x \in (-1, 1)$. Even though the functions f_n are continuous on A, their limit f is not continuous at $x = 1$. This fact can be expressed as

$$\lim_{x\to 1} \lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty} \lim_{x\to 1} f_n(x).$$

Example 2. Let $f_n(x) = n^2 x(1-x)^n$. Then $f(x) = \lim_{n\to\infty} f_n(x) = 0$ on $[0,1]$. However,

$$\int_0^1 f_n(x)\,dx = n^2 \int_0^1 (1-x)x^n\,dx = \frac{n^2}{n+1} - \frac{n^2}{n+2} = \frac{n^2}{(n+1)(n+2)}$$

so that

$$1 = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0.$$

These examples point out that one cannot interchange limits in general; note that an integral is also a limit. In certain cases this can be remedied with a stronger notion of convergence, namely uniform convergence.

Definition. A sequence of functions (f_n) on a set $S \subseteq \mathbb{R}$ is said to converge uniformly to a function f on S if, for every $\epsilon > 0$, there is an N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for every $x \in S$.

Note that the N is independent of x, which is not necessarily the case for pointwise convergence.

Example 3. The sequence of functions (f_n) in Example 1 does not converge uniformly on $S = (-1, 1]$. In fact, it does not converge uniformly on $(0, 1)$. To see this have to show that

$$(\exists \epsilon > 0)(\forall N)(\exists x, 0 < x < 1)(\exists n \geq N)|f_n(x) - f(x)| \geq \epsilon.$$

Since $0 < x < 1$, we have $f_n(x) - f(x) = x^n$. Let $\epsilon = 1/2$. Then we have to show that, given N, we have $x^n > 1/2$ for some x and some $n \geq N$. To do this pick x so that

$$1 > x > \frac{1}{\sqrt[N]{2}}.$$

This is possible since $\sqrt[N]{2} > 1$. Then $0 < x < 1$ and $x^N \geq 1/2$.

Example 4. The sequence of functions (f_n) in Example 1 converges uniformly on $[0, a]$ for any $0 < a < 1$. Indeed, given $\epsilon > 0$, choose N such that $a^N < \epsilon$. Then $0 \leq x \leq a$ and $n \geq N$ implies $x^n \leq a^n \leq a^N < \epsilon$

Exercise 1. Prove that the sequence of functions in Example 2 does not converge uniformly.

Theorem. Let (f_n) be a sequence of functions defined on a set S. Then

$$(f_n) \text{ converges uniformly on S} \iff (\forall \epsilon > 0)(\exists N)(\forall m, n \geq N)(\forall x \in S)|f_m(x) - f_n(x)| < \epsilon.$$

This is the **Cauchy Condition** for uniform convergence. The proof is left as an exercise.

Definition. If $(f_n)_{n\geq 1}$ is a sequence of functions on the set S, the series $\sum_{n=1}^{\infty} f_n$ is said to converge uniformly to f on S if the sequence (s_n) of partial sums, defined by

$$s_n(x) = \sum_{k=1}^{n} f_k(x),$$

converge uniformly to f on S. It converges absolutely on S if $\sum_{n=1}^{\infty} |f_n|$ converges on S.

Theorem. The infinite series $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on S if and only if

$$(\forall \epsilon > 0)(\exists N)(\forall x \in S)(\forall m, n \geq N, m < n) \quad \left|\sum_{k=m}^{n} f_k(x)\right| < \epsilon$$

Corollary (Weierstrass M-Test). If $|f_n(x)| \leq M_n$ for all $x \in S$ and $\sum M_n$ converges, then $\sum f_n$ is uniformly and absolutely convergent on S.

Proof. We have $|\sum_{k=m}^{n} f_k(x)| \leq \sum_{k=m}^{n} |f_k(x)| \leq \sum_{k=m}^{n} M_k$ for all $x \in S$.

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