MATH 255: Lecture 10

The Riemann Integral: Lebesgue's Integrability Criterion

**Definition.** A set $S$ of real numbers is said to have measure zero if, for every $\epsilon > 0$, the set $S$ is contained in a countable union of intervals, the sum of whose lengths is less than $\epsilon$.

**Theorem.** If $S_1, S_2, \ldots, S_n, \ldots$ are each of measure zero then their union is also of measure zero.

**Proof.** Each set $S_k$ can be covered by a countable union of intervals, the sum of whose lengths $< \epsilon/2^k$. The union of all the intervals so obtained is also countable and the sum of the lengths is less than $\sum_1^\infty 1/2^k = \epsilon$.

**QED**

A countable set is of measure zero but there are uncountable sets of measure zero. For example, the Cantor set consisting of all the real numbers in the interval $[0,1]$ whose representation to the base 3 contain only 0 or 2, is an uncountable set of measure zero. The proof of this is left as an exercise.

**Definition.** Let $f$ be a bounded function defined on a subset $S \subseteq \mathbb{R}$. The oscillation of $f$ on $S$ is

$$\Omega_f(S) = \sup_{x,y \in S} |f(x) - f(y)|.$$

If $T \subseteq S$, we have $\Omega_f(T) \le \Omega_f(S)$. Let $N_r(c) = \{x \mid |x - c| < r\}$.

**Definition.** If $f$ is a bounded function on $S$ and $c \in S$, the oscillation of $f$ at $c$ is defined to be

$$\omega_f(c) = \inf_{r>0} \Omega_f(S \cap N_r(c)).$$

**Exercise 1.** If $f$ is a bounded function on $S$ and $c \in S$, then $f$ is continuous at $c \iff \omega_f(c) = 0$.

**Theorem (Lebesgue's Integrability Criterion).** A bounded function on $[a,b]$ is Riemann integrable if and only if the points of discontinuity of $f$ form a set $D$ of measure zero.

**Proof.** ($\Rightarrow$) Let $\epsilon > 0$ be given and let $D_i$ be the set of points $x$ with $\omega_f(x) \ge \epsilon/2^i$. Let $P$ be the partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ with

$$U(P,f) - L(P,f) = \sum_{k=1}^n (M_k - m_k)\Delta x_k < \frac{\epsilon}{4^i}.$$

If $x \in D_i \cap (x_{k-1}, x_k)$ there is an $r > 0$ such that $N_r(x) \subseteq (x_{k-1}, x_k)$ so that

$$\frac{\epsilon}{2^i} \le \omega_f(x) \le \Omega_f(N_r(x)) \le M_k - m_k.$$

If $T$ is the set of these $k$ with $D_i \cap (x_{k-1}, x_k) \ne \emptyset$, it follows that

$$\frac{\epsilon}{2^i} \sum_{k \in T} \Delta x_k \le \sum_{k=1}^n (M_k - m_k)\Delta x_k < \frac{\epsilon}{4^i}.$$

Hence $\sum_{k\in T}[x_k - x_{k-1}] < \epsilon/2^i$ and $D_i \subseteq \bigcup_{k\in T}[x_{k-1}, x_k]$. This shows that each $D_i$ is contained in the union of a finite number of intervals, the sum of whose lengths is less than $\epsilon/2^i$. Since $D = \bigcup D_i$, it follows that $D$ is of measure zero.

($\Leftarrow$) Let $M > 0$ be an upper bound for $|f|$ on $[a,b]$ and let $\epsilon > 0$ be given. Since $D$ is of measure zero, it can be covered by open intervals $J_i$, $(i \ge 1)$, the sum of whose lengths is less that $\epsilon/4M$. We now define a function $\delta$ on $[a,b]$ as follows:

1. If $t \in D$, there is a $k$ such that $t \in J_k$. Thus there is a $\delta(t) > 0$ such that $N_{\delta(t)}(t) \subseteq J_k$.
2. If $t \notin D$, there is a $\delta(t) > 0$ such that $x \in N_{\delta(t)}(t) \Rightarrow |f(x) - f(t)| < \epsilon/4(b-a)$.

**Lemma.** If $\delta$ is a function on $[a, b]$ such that $\delta(x) > 0$ for all $x$, there is a partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of $[a, b]$ and a tag $t$ for $P$ such that for all $k$ we have $[x_{k-1}, x_k] \subseteq N_{\delta(t_k)}(t_k)$.

We call such a tagged partition **$\delta$-fine** with **gauge** $\delta$.

**Proof.** Let $S$ be the set of those $x \in [a, b]$ such that there is a $\delta$-fine tagged partition of $[a, x]$. The set $s$ is not empty since, for any $c \in [a, b]$ with $a < c < a + \delta(a)$, the partition $\{a, c\}$ with tag $a$ of $[a, c]$ is $\delta$-fine. If $x \in S$ and $x < b$, we can choose $c$ so that $x < c < \min(b, x + \delta(x)$. If $(P, t)$ is a $\delta$-fine partition of $[a, x]$, then $(P \cup \{c\}, (t, c))$ is a $\delta$-fine tagged partition of $[a, c]$. This shows that $\sup S = b$. To show that $b \in S$, choose $c \in S$ so that $\max(a, b - \delta(b)) < c < b$. If $(P, t)$ is a $\delta$-fine partition of $[a, c]$, then $(P \cup \{b\}, (t, b))$ is a $\delta$-fine tagged partition of $[a, b]$. **QED**

Let $(P, t)$ be a $\delta$-fine tagged partition of $[a, b]$ and consider

$$U(P, f) - L(P, f) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k = \sum_{k \in A} (M_k - m_k)\Delta x_k + \sum_{k \in B} (M_k - m_k)\Delta x_k,$$

where $A$ is the set of the $k$ with $t_k \in D$ and $B$ the set of those $k$ with $t_k \notin D$. If $k \in A$, we have $M_k - m_k \leq 2M$ and we have $M_k - m_k < \epsilon/2(b - a)$ if $k \in B$. The sum of the lengths of the intervals $J_{k,i} = J_k \cap [x_{k-1}, x_k]$ with $k \in A$ and $i \geq 1$ is less than $\epsilon/4M$ and the sum of the lengths of the intervals for a fixed $k \in A$ is $\geq \Delta x_k$. Since $J_{k,i} \cap J_{\ell,j} = \emptyset$ for $k \neq \ell$, it follows that $\sum_{k \in A} \Delta x_k$ is less than the sum of the lengths of the intervals $J_{k,i}$. Hence,

$$\sum_{k \in A} (M_k - m_k)\Delta x_k \leq 2M \sum_{k \in A} \Delta x_k < 2M \frac{\epsilon}{4M} = \frac{\epsilon}{2}, \qquad \sum_{k \in B} (M_k - m_k)\Delta x_k < \frac{\epsilon}{2(b - a)} \sum_{k \in B} \Delta x_k \leq \frac{\epsilon}{2}$$

which gives $U(P, f) - L(P, f) < \epsilon$. Since $\epsilon > 0$ is arbitrary, $f \in \mathcal{R}(a, b)$. **QED**