MATH 255: Assignment 9

(due Wednesday, April 2)

1. (a) If $\sum a_n/n^p$ converges and $p < q$, show that $\sum a_n/n^q$ converges.
   (b) Let $\sum_{n=1}^{\infty} a_n$ be a strictly positive series and let $r_n = \sum_{k=n+1}^{\infty} a_k$. If

$$R_n = n(1 - a_{n+1}/a_n) \geq a > 1 \text{ for } n \geq N,$$

   show that $r_n \leq n a_{n+1}/(a-1)$ for $n \geq N$. (Remainder estimate for Raabe's test).

2. If a, b, c are not zero or negative integers, find the interval of convergence of the hypergeometric series

$$1 + \frac{ab}{1!c}x + \frac{a(a+1)(b(b+1)}{2!c(c+1)}x^2 + \cdots + \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}x^n + \cdots .$$

3. (a) If $|a_n| = O(n^c)$, show that the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ converges absolutely and uniformly for $s \geq c + 1 + \epsilon$ for any $\epsilon > 0$. What can you say about the function defined by the Dirichlet series when $s > c + 1$?
   (b) If $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, show that

$$(1 - \frac{1}{2^{s-1}})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

   Prove that $F(s) = \sum_{n=1}^{\infty}(-1)^{n-1}/n^s$ is continuous for $s > 0$. Deduce that $\lim_{s\to 1+} \zeta(s) = \infty$.

4. Show that the function

$$f(s) = \int_1^{\infty} \frac{\sin x}{x^s}\, dx$$

   is continuous for $s > 0$.

REMINDER: There will be a 10 minute theory test on Friday March 28 covering the material in lectures 7,8.