MATH 255: Assignment 4

(due Monday, February 10)

1. (a) If f is a bounded function on $[a, b]$ such that f is Riemann integrable on $[c, b]$ for all $a < c < b$, prove that f is Riemann integrable on $[a, b]$.

   (b) If f is the function on $[0, 1]$ defined by $f(0) = 0$ and $f(x) = \sin(1/x)$ if $x \neq 0$, prove that $f \in \mathcal{R}(0, 1)$.

2. Which of the following functions f are of bounded variation on $[0, 1]$?

   (a) $f(0) = 0$, $f(x) = x^2 \sin(1/x)$ if $x \neq 0$;

   (b) $f(0) = 0$, $f(x) = \sqrt{x} \sin(1/x)$ if $x \neq 0$.

   Justify your answers.

3. Let α, f be functions on $[a, b]$ with α is strictly increasing and f is continuous. If $f(x) \geq 0$ on $[a, b]$, show that

$$\int_a^b f(x)\, d\alpha(x) = 0 \implies f = 0.$$

4. Let $f \in \mathcal{R}(a, b)$ and let $g(x) = \int_a^x f(t)\, dt$ if $x \in [a, b]$. Prove that the total variation of g on $[a, x]$ is equal to $\int_a^x |f(t)|\, dt$.