MATH 255: Assignment 10

(due Friday, April 11)

1. Show that the remainder after n terms of the series

$$\frac{\pi}{2} = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} + \cdots + \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot \frac{1}{2n+1} + \cdots .$$

is at most $2/\sqrt{n+2}$ for $n \geq 10$.

2. Let S be the set of real-valued functions on $[a, b]$ which are continuously differentiable. Show that

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| + |f'(x) - g'(x)|$$

is a metric on $[a, b]$ and that S is complete with respect to this metric.

3. (a) Show that the continuous image of a connected set is connected.
   (b) Show that $X \subseteq \mathbb{R}$ is connected if and only if X is a interval. Show how the intermediate value theorem for continuous real-valued functions on $\mathbb{R}$ follows from this and (a).
   (c) Show that $\mathbb{R}^n$ is connected.

4. Let $f(x) = \left(\int_0^x e^{-t^2} \, dt \right)^2, \quad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1} \, dt.$
   (a) Show that $g'(x) + f'(x) = 0$ and deduce that $g(x) + f(x) = \pi/4$.
   (b) Use (a) to prove that

$$\int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2}.$$