McGill University
MATH 251: Algebra 2
Solutions to Assignment 8

1. Since $P^2 - P = 0$ we have $V = \mathrm{Ker(P^2-P)} = \mathrm{Ker(P(P-1))} = \mathrm{Ker(P)} \oplus \mathrm{Ker(P-1)} = \mathrm{Ker(P)} \oplus \mathrm{Ker(Q)}$. Now $v \in \mathrm{Ker(Q)} \implies \mathrm{v = P(v)} \implies \mathrm{v \in Im(P)}$ and $v \in \mathrm{Im(P)} \implies \mathrm{v = P(u)} \implies \mathrm{Q(v)} = \mathrm{(1-P)P(u)} = 0 \implies \mathrm{v \in Ker(Q)}$. Hence $\mathrm{Ker(Q) = Im(P)}$. Similarly, since $Q^2 = (1-P)^2 = 1 - 2P + P^2 = 1 - P = Q$, we have $\mathrm{Ker(P) = Im(Q)}$. If V is an inner product space, we have $P = P_W \iff \mathrm{Im(P) \perp Ker(P)}$. If P is self-adjoint then $P(u) = 0 \implies <u, P(v)> = <P(u), v> = 0$ which implies $P = P_W$. Conversely, if $P = P_W$ then $<P(u), v> = <P(u), P(v) + Q(v)> = <P(u), P(v)> = <P(u) + Q(u), P(v)> = <u, P(v)>$ which implies that P is self-adjoint.

2. (a) If $A = [a_{ij}], B = [b_{ij}]$ we have $<A, B> = \mathrm{tr(AB^t)} = \sum_{\mathrm{i,j}} \mathrm{a_{ij} b_{ij}} = \mathrm{tr(BA^t)} = \mathrm{<B, A>}$. Hence $<A, A> = \sum_{ij} a_{ij}^2 \geq 0$ with equality iff $a_{ij} = 0$ for all i, j. We also have $<aX + bY, Z> = \mathrm{tr((aX + bY)Z^t)} = \mathrm{tr(aXZ^t + bYZ^t)} = \mathrm{a <X, Z> + b <X, Z>}$ so that $<,>$ is an inner product.

   (b) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ we have $<X, T(Y)> = \mathrm{tr(X(AY)^t)} - \mathrm{tr(X(YA)^t)} = \mathrm{tr(XY^tA)} - \mathrm{tr(XAY^t)} = \mathrm{tr(AXY^t)} - \mathrm{tr(XAY^t)} = \mathrm{<T(X), Y>}$. Since $T^3 = 4T$ we have $T(T-2)(T+2) = 0$ so that the possible eigenvalues are $0, 2, -2$. We have $\mathrm{Ker(T)} = \mathrm{Span}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})$, $\mathrm{Ker(T-2)} = \mathrm{Span}(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix})$, $\mathrm{Ker(T+2)} = \mathrm{Span}(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix})$. Applying Gram-Schmidt to each eigenspace and normalizing we get the following orthonormal basis of V

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

   consisting of eigenvectors of T.

3. (a) We have $(A - I)^2 = 3(A - I)$ so that $(A - I)(A - 4I) = 0$. Since $A \neq I, 4I$, the polynomial $(X - 1)(X - 4)$ is the minimal polynomial of A and $1, 4$ are the eigenvalues of A. We have

$$\mathrm{Null(A - I)} = \mathrm{Span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathrm{Null(A - 4I)} = \mathrm{Span}(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}).$$

   Applying Gram-Schmidt to the second eigenspace and normalizing, we get

$$f_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad f_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad f_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

   The matrix P with columns f_1, f_2, f_3 is an orthogonal matrix with

$$P^t AP = P^{-1} AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (b) If $X = PY$ we have $||X||^2 = <PY, PY> = <Y, P^t PY> = <Y, Y> = ||Y||^2$ so that $||X|| = ||Y||$. Since $q(X) = X^t AX = Y^t P^t APY = 4y_1^2 + y_2^2 + y_3^2$ we have

$$||X||^2 = ||Y||^2 = y_1^2 + y_2^2 + y_3^2 \leq q(X) \leq 4y_1^2 + 4y_2^2 + 4y_2^2 = 4||Y||^2 = 4||X||^2.$$

   Hence $1 \leq q(X) \leq 4$ on the unit sphere $||X|| = 1$.

(c) To obtain the spectral decomposition of A we have $A = 4f_1f_1^t + f_2f_2^t + f_3f_3^t$. Hence

$$A = 4\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1/6 & 1/6 & -1/3 \\ 1/6 & 1/6 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

Adding the last two matrices yields the spectral decomposition

$$A = 4\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

(d) The following eight matrices B are symmetric and $B^2 = A$:

$$\begin{aligned} B &= \pm 2\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \pm \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \pm \begin{bmatrix} 1/6 & 1/6 & -1/3 \\ 1/6 & 1/6 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \\ &= \pm\begin{bmatrix} 4/3 & 1/3 & 1/3 \\ 1/3 & 4/3 & 1/3 \\ 1/3 & 1/3 & 4/3 \end{bmatrix}, \pm\begin{bmatrix} 1/3 & 4/3 & 1/3 \\ 4/3 & 1/3 & /3 \\ 1/3 & 1/3 & 4/3 \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

To see that these are the only possible such symmetric matrices B we use the fact that B commutes with A. This implies that A and B are simultaneously diagonalizable. Thus, there is an invertible matrix Q with

$$Q^{-1}AQ = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^{-1}BQ = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Since $B^2 = A$, we have $a^2 = 4, b^2 = c^2 = 1$ which gives $a = \pm 2, b = c = \pm 1$. This yields precisely 8 matrices B.

4. (a) We have $A^tA = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. The matrix $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is an orthogonal matrix whose columns f_1, f_2 are eigenvectors of A with eigenvalues $2, 4$ respectively. We have

$$Af_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{2}g_1$$

$$Af_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix} = 2\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = 2g_2.$$

The vectors g_1, g_2 defined above are orthonormal vectors and $\sqrt{2}, 2$ are the singular values of A. The singular value decomposition of A is

$$\begin{aligned} A &= \sqrt{2}g_1f_1^t + 2g_2f_2^t \\ &= \sqrt{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + 2\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \sqrt{2}\begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & -1\sqrt{2} \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}. \end{aligned}$$

(b) The generalized inverse of A is

$$A^+ = \frac{1}{\sqrt{2}}f_1g_1^t + \frac{1}{2}f_2g_2^t = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}.$$

5. (a) We have $AA^* = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \begin{bmatrix} -i & 1 \\ 1 & -ii \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ 1 & -ii \end{bmatrix} \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} = A^*A$ so A is normal.

(b) Since A has constant row sums $= 1+i$ the the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $1+i$. Since the trace of A is the sum of the eigenvalues, the other eigenvalue is $i-1$ with eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The matrix $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, whose columns f_1, f_2 are the normalized eigenvectors of A, is a unitary matrix with $U^{-1}AU = \begin{bmatrix} 1+i & 0 \\ 0 & i-1 \end{bmatrix}$.

(c) The spectral decomposition of A is

$$A = (1+i)f_1 f_1^* + (i-1)f_2 f_2^* = (1+i) \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + (i-1) \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$