McGill University
MATH 251: Algebra 2
Solutions to Assignment 7

1. (a) $\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_2-3R_1 \\ R_3-2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -10 & -5 & -3 & 1 & 0 \\ 0 & -5 & -3 & -2 & 0 & 1 \end{array}\right] \begin{array}{c} C_2-3C_1 \\ C_3-2C_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -10 & -5 & -3 & 1 & 0 \\ 0 & -5 & -3 & -2 & 0 & 1 \end{array}\right]$

$R_3-\frac{1}{2}R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -10 & -5 & -3 & 1 & 0 \\ 0 & 0 & -1/2 & -1/2 & -1/2 & 1 \end{array}\right] C_3-\frac{1}{2}C2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -10 & 0 & -3 & 1 & 0 \\ 0 & 0 & -1/2 & -1/2 & -1/2 & 1 \end{array}\right] \begin{array}{c} \frac{1}{\sqrt{10}}R_2 \\ \sqrt{2}R_3 \end{array}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1/\sqrt{10} & 0 & -3/\sqrt{10} & 1/\sqrt{10} & 0 \\ 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & \sqrt{2} \end{array}\right] \begin{array}{c} \frac{1}{\sqrt{10}}C_2 \\ \sqrt{2}C_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3/\sqrt{10} & 1/\sqrt{10} & 0 \\ 0 & 0 & -1 & -1/\sqrt{2} & -1/\sqrt{2} & \sqrt{2} \end{array}\right]$

Hence $PAP^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ where $P = \begin{bmatrix} 1 & 0 & 0 \\ -3/\sqrt{10} & 1/\sqrt{10} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & \sqrt{2} \end{bmatrix}$.

(b) The quadratic form $q(x) = xAx^t$ has rank $1+2=3$ and signature $1-2=-1$. If $x = yP$ then $y = xP^{-1}$ and $q(x) = yPAP^ty^t = y_1^2 - y_2^2 - y_3^2$. Since

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & \sqrt{10} & 0 \\ 2 & \sqrt{10}/2 & 1/\sqrt{2} \end{bmatrix}$$

we have $y_1 = x_1 + 3x_2 + 2x_3, y_2 = \sqrt{10}x_2 + (\sqrt{10}/2)x_3, y_3 = (1/\sqrt{2})x_3$.

(c) The rows of P are the required vectors.

2. $\left[\begin{array}{ccc|ccc} 1 & i & -i & 1 & 0 & 0 \\ -i & 2 & 1-i & 0 & 1 & 0 \\ i & 1+i & 7 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_2+iR_1 \\ R_3-iR_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & i & -i & 1 & 0 & 0 \\ 0 & 1 & 2-i & i & 1 & 0 \\ 0 & 2+i & 6 & -i & 0 & 1 \end{array}\right] \begin{array}{c} C_2-iC_1 \\ C_3+iC_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2-i & i & 1 & 0 \\ 0 & 2+i & 6 & -i & 0 & 1 \end{array}\right]$

$R_3-(2+i)R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2-i & i & 1 & 0 \\ 0 & 0 & 1 & 1-3i & -2-i & 1 \end{array}\right] C_3-(2-i)C2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & i & 1 & 0 \\ 0 & 0 & 1 & 1-3i & -2-i & 1 \end{array}\right]$

We have $q(x) = xA\overline{x}^t = yPA\overline{P}^ty = y_1^2 + y_2^2 + y_3^2$ where $x = Py$, $y = xP^{-1}$ and

$$A = \begin{bmatrix} 1 & i & -i \\ -i & 2 & 1-i \\ i & 1+i & 7 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ i & 1 & 0 \\ 1-3i & -2-i & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -i & 1 & 0 \\ i & 2+i & 1 \end{bmatrix}$$

so that $y_1 = x_1 - ix_2 + ix_3, y_2 = x_2 + (2+i)x_3, y_3 = x_3$, $f_1 = e_1, f_2 = ie_1 + e_2, f_3 = (1-3i)e_1 - (2+i)e_2 + e_3$.

3. The first step is to orthogonally project $Y = [1,2,3,2]^t$ onto the column space of the coefficient matrix A. An orthogonal basis for the column space of A is $C_1 = [1,2,1,1]^t, C_2 = [4,1,4,-10]^t$. The orthogonal projection of Y on the column space of A is

$$B = \frac{< Y, C_1 >}{< C_1, C_1 >}C_1 + \frac{< Y, C_2 >}{< C_2, C_2 >}C_2 = \frac{10}{7}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \frac{-2}{133}\begin{bmatrix} 4 \\ 1 \\ 4 \\ -10 \end{bmatrix} = \frac{2}{133}\begin{bmatrix} 91 \\ 189 \\ 91 \\ 105 \end{bmatrix}.$$

The second step is to find a solution of $AX = B$. We find that $X = [28/19, -2/19, 0]^t$ is a solution. The least squares solution of smallest norm is the orthogonal projection Z of this solution X onto the column space of

$$A^t = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 3 & 1 & 5 \end{bmatrix}.$$

An orthogonal basis for the column space of A^t is $D_1 = [1,1,1]^t, D_2 = [-1,-4,5]^t$. The orthogonal projection of X on the column space of A^t is

$$Z = \frac{26}{57}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{10}{399}\begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix} = \frac{1}{133}\begin{bmatrix} 64 \\ 74 \\ 44 \end{bmatrix}.$$

4. An orthogonal basis for W is $f_1(x) = 1$, $f_2(x) = x - \frac{1}{2}$, $f_3(x) = x^2 - x + \frac{1}{6}$. The best approximation to $h(x) = x^3$ by a function in W is the orthogonal projection of h on W. This is the function

$$\begin{aligned} h(x) &= \frac{< h, f_1 >}{< f_1, f_1 >} f_1 + \frac{< h, f_2 >}{< f_2, f_2 >} f_2 + \frac{< h, f_3 >}{< f_3, f_3 >} f_3 \\ &= \frac{1}{4} + \frac{9}{10}(x - \frac{1}{2}) + \frac{3}{2}(x^2 - x + \frac{1}{6}) \\ &= \frac{3}{2}x^2 - \frac{3}{5}x + \frac{1}{20}. \end{aligned}$$