McGill University
MATH 251: Algebra 2
Assignment 6 Solutions

1. (a) Since $A^2 = nA$, we have $A(A - nI) = 0$ so that the possible eigenvalues of A are $0, n$.
If the characteristic of F does not divide n, the minimal polynomial of A is $X - 1$ if $n = 1$ and $X(X - n)$ is $n > 1$ since, in this case, $A, A - nI \neq 0$. Hence, A is diagonalizable. If $n = 1$ then P is already diagonal and we may take $P = I$. If $n > 1$ then

$$P^{-1}AP = \begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{with} \quad P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

since $\mathrm{Null(A - nI) = Span(e_1 + e_2 + \cdots + e_n)}$ and $\mathrm{Null(A) = Span(e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n)}$.
If the characteristic of F divides n then the minimal polynomial of A is X^2. Since $\mathrm{Null(A) = Span(e_1 - e_2, \ldots, e_1 - e_n)}$ and $Ae_1 = e_1 + e_2 + \cdots + e_n$ we see that there is one Jordan block of size 2 for the eigenvalue 0 with cyclic vector e_1 and $n-2$ Jordan blocks of size 1 for the eigenvalue 0 with cyclic vectors $e_1 - e_3, \ldots e_1 - e_n$. Hence

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{with} \quad P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

(b) If $c_1, c_2, \ldots, c_m$ are the non-zero eigenvalues of T and $T(u_i) = c_i u_1$ with $u_i \neq 0$ then $u_1, u_2, \ldots, u_n \in \mathrm{Im(T)}$ and are linearly independent since they are eigenvectors of T with distinct eigenvalues. Hence $m \leq n = \mathrm{rank(T)}$. Since the only additional eigenvalue that T can have is the zero eigenvalue, we have $m \leq n + 1$.

2. (a) Since T is diagonalizable, its minimum polynomial $m_T(X)$ is a product of distinct linear factors. If R is the restriction of T to a T-invariant subspace W then $m_T(R) = 0$ since $m_T(T) = 0$. Hence $m_R(X)$ is a product of distinct linear factors since $m_R(X)$ divides $m_T(X)$. Thus R is diagonalizable.

(b) If $T(u) = cu$ then $S(T(u)) = cS(u)$ so that $T(S(u)) = cS(u)$ since $ST = TS$. Thus $u \in \mathrm{Ker(T - c)} \implies \mathrm{S(u) \in Ker(T - c)}$ and hence $\mathrm{Ker(T - c)}$ is S-invariant.

(c) Since the restriction of S to each eigenspace of T is diagonalizable, a basis for each eigenspace of T can be found consisting of eigenvectors of S. Since V is the direct sum of the eigenspaces of T, the union of these bases is a basis of V consisting of vectors which are simultaneously eigenvectors of S and T.

3. We have $N_1 = \mathrm{Null(A - I) = Span(e_1, e_2, e_3)}$, $N_2 = \mathrm{Null((A - I)^2) = Span(e_1, e_2, e_3, e_4, e_6)}$, $N_3 = \mathrm{Null((A - I)^3)} = \mathbb{F}^{6\times 1}$ so that $\dim(N_1) = 3$, $\dim(N_2) = 5$, $\dim(N_3) = 6$ and we have

$$N_3/N_2 = \mathrm{Span}(\overline{\mathrm{e}}_5), \quad \mathrm{N_2/N_1 = Span(\overline{e}_4, \overline{e}_6)}.$$

Thus 1 is the only eigenvalue and there is exactly one Jordan block of size 3 with cyclic generator e_5. Since The image of $(A - I)e_5 = e_1 + e_2 + e_4$ in N_2/N_1 is $\overline{e}_4$ we see that there is one Jordan block of size 2 with cyclic generator e_6. Since $(A - I)^2 e_5 = e_1 + e_2$ and $(A - I)e_6 = e_1 + e_2 + e_3$, we see that there is 1 Jordan block of size 1 with cyclic generator e_2. If P is the 6×6 matrix whose columns are $e_2, (A - I)e_6, e_6, (A - I)^2 e_5, (A - I)e_5, e_5$ we have

$$P = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. (a) We have $\mathrm{Null(B-I) = Span(e_1 - 2e_2 + e_3)}$, $\mathrm{Null((B-I)^2) = Span(e_2, e_1 + e_3) = Null((B-I)^3)}$, $\mathrm{Null(B-2I) = Span(e_1 - 2e_2) = Null((B-2I)^2)}$ which implies that the Jordan canonical form of B has 1 Jordan block of size 1 for the eigenvalue 2 and 1 Jordan block of size 2 for the eigenvalue 2 with cyclic generators $e_2, e_1 - 2e_2$ respectively. Hence A, B are similar and $P^{-1}BP = A$ with

$$P = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) The matrices A, B are not similar since $\mathrm{rank(A - I) = 1}$, $\mathrm{rank(B - I) = 2}$.

(c) Let $W_i = \mathrm{Span(e_1, \ldots, e_i)}$ for $0 \le i \le n$. Then, for $2 \le i \le n$, we have $(A - cI)e_i = a_{(i-1)i}e_{i-1} + f_i$ with $f_i \in W_{i-2}$. Since $(A - cI)(W_i) \subseteq (W_{i-1})$, it follows that $(A - cI)^{n-1}e_n = a_{12}a_{23}\cdots a_{(n-1)n}e_1 \neq 0$ and $(A - cI)^n = 0$. Thus the Jordan canonical form of A has one Jordan block of size n for the eigenvalue c.