McGill University
MATH 251: Algebra 2
Assignment 5 Solutions

1. (a) Since $a(u,0)+b(v,0)=(au+bv,0)$ and $a(0,u)+b(0,v)=(0,au+bv)$ we see that U_1,V_1 are subspaces since $(0,0)\in U_1,V_1$. It also follows that the mappings $f_1:U\to U_1$, $f_2:V\to V_1$ defined by $f_1(u)=(u,0)$, $f_2(v)=(0,v)$ are linear and hence isomorphisms since they are bijective. Since $(u,v)=(u,0)+(0,v)$ we see that $V\times V$ is the sum of U_1 and V_1. The sum is direct since $U_1\cap V_1=\{(0,0)\}$.

   (b) Since $T(a(u_1,v_1)+b(a_2,v_2))=T(au_1+bu_2,av_1+bv_2)=(au_1+bu_2)+(av_1+bv_2)=a(u_1+v_1)+b(v_1+v_2)=aT(u_1,v_1)+bT(u_2,v_2)$ we see that T is linear. The image of T is $U+V$ and the kernel of T is $Z=\{(u,u)\mid u\in U\cap V\}$. The mapping $g:U\cap V\to Z$ defined by $g(u)=(u,u)$ is linear since $g(au+bv)=(au+bv,au+bv)=a(u,u)+b(v,v)=ag(u)+bg(v)$. Since g is bijective, it is an isomorphism. Hence $\dim(U+V)+\dim(U\cap V)=\dim(\mathrm{Im(T)}+\dim(\mathrm{Ker(T)})=\dim(\mathrm{U\times V})=\dim(\mathrm{U})+\dim(\mathrm{V})$.

   (c) Let $u_1,u_2,\dots,u_r$, $v_1,v_2,\dots,v_s$, $w_1,\dots,w_t$ be distinct elements of B_1, B_2, B_3 respectively. If

$$a_1u_1+\ldots+a_ru_r+b_1v_1+\cdots+b_sv_s+c_1w_1+\cdots+w_t=0$$

$a_1u_1+\ldots+a_ru_r+b_1v_1+\cdots+b_sv_s=c_1w_1+\cdots+w_t\in U\cap V$ which implies $c_1=\cdots c_t=0$. But then

$$a_1u_1+\ldots+a_ru_r+b_1v_1+\cdots+b_sv_s=0$$

which implies $a_1=\cdots=a_r=b_1=\cdots=c_s=0$. Hence $B_1\cup B_2\cup B_3$ is independent and hence a basis of $U+V$ since $U+V$ is spanned by this set.

2. (a) If $A=[a_{ij}],B=[b_{ij}]$ then $\mathrm{tr(a\,A+b\,B)}=\sum_{\mathrm{i}}(\mathrm{a\,a_{ii}+bb_{ii}})=\mathrm{a}\sum_{\mathrm{i}}\mathrm{a_{ii}}+\mathrm{b}\sum_{\mathrm{i}}\mathrm{b_{ii}}=\mathrm{a\,tr(A)+b\,tr(B)}$ and $\mathrm{tr(AB)}=\sum_{\mathrm{i,j}}\mathrm{a_{ij}b_{j\,i}}=\sum_{\mathrm{i,j}}\mathrm{b_{ij}a_{j\,i}}=\mathrm{tr(BA)}$.

   (b) If E_{ij} is the $n\times n$ matrix with (i,j)-th entry 1 and all other entries equal to 0, we have $E_{ij}E_{k\ell}=\delta_{jk}E_{i\ell}$. Hence $E_{ik}E_{kj}-E_{kj}E_{ik}=E_{ij}$ if $i\neq j$ and $E_{ii}-E_{kk}$ if $i=j$. If $W=\mathrm{Ker(tr)}$ then $\dim(W)=n^2-1$. Since the n^2-n matrices E_{ij} with $i\neq j$ together with the $n-1$ matrices $E_{11}-E_{jj}$ with $2\leq j\leq n$ are a basis for W, we see that W is spanned by the matrices of the form $AB-BA$. If ϕ is a linear form on $F^{n\times n}$ then, since every matrix $A=[a_{ij}]$ can be written in the form $\mathrm{tr(A)E_{11}+B}$ with $B\in W$, we have that $\phi(A)=\phi(E_{11})\mathrm{tr(A)}$ which shows that $\phi=c\,\mathrm{tr}$ with $c=\phi(E_{11})$.

   (c) We have $B=P^{-1}AP$ so that $\mathrm{tr(B)=tr(P^{-1}AP)=tr(APP^{-1})=tr(A)}$.

3. (a) If $W=\mathrm{Span}((1,1,1,1,1),(1,0,1,1,0),(0,1,0,1,0))$ and $\phi(x_1,\dots,x_5)=a_1x_1+\ldots+a_5x_5$ then $\phi\in W^0$ if and only if $a_1+a_2+a_3+a_4+a_5=0$, $a_1+a_3+a_4=0$, $a_2+a_4=0$. Since the solution set of this system of equations has for basis $(1,0,1,0,0),(1,1,0,1,1)$, the subspace W^0 has for basis ϕ_1,ϕ_2, where $\phi_1(x)=x_1+x_3$, $\phi_2(x)=x_1+x_2+x_4+x_5$.

   (b) We have $\phi\in(U+V)^0\iff\phi(U)=\phi(V)=\{0\}\iff\phi\in U^0\cap V^0$. If V is finite-dimensional we have $(U^0+V^0)^0=U^{00}+V^{00}=U+V$ and $(U\cap V)^{00}=U\cap V$ which implies $(U\cap V)^0=U^0+V^0$ since they have the same annihilator.

4. (a) We have $\phi\in\mathrm{Im(T)^0}\iff\phi\circ\mathrm{T(U)}=0\iff\mathrm{T^t}(\phi)(\mathrm{U})=0\iff\mathrm{T^t}(\phi)=0\iff\phi\in\mathrm{Ker(T^t)}$ and $u\in\mathrm{Im(T^t)^0}\iff\phi(\mathrm{T(u)})=0$ for all $\phi\in\mathrm{V^*}\iff\mathrm{T(u)}=0\iff\mathrm{u\in Ker(T)}$

   (b) If $\dim(U)<\infty$ we have $\mathrm{rank(T)=dim(Im(T))=dim(U)-dim(Ker(T))=dim(Im(T^t))=rank(T^t)}$ and if $\dim(V)<\infty$ we have $\mathrm{rank(T^t)=dim(Im(T^t))=dim(V)-dim(ker(T^t))=dim(Im(T))=rank(T)}$.