McGill University
MATH 251: Algebra 2
Assignment 4 Solutions

1. The function $y = f(xf$ is a solution of the given differential equation $\iff$ $f \in \mathrm{Ker}(D^4 - 6D^3 + 13D^2 - 12D + 4)$, where D is the differentiation operator on the vector space of infinitely differentiable real-valued functions on $\mathbb{R}$. Since $X^4 - 6X^3 + 13X^2 - 12X + 4 = (X-1)^2(X-2)^2$, we have
$$\begin{aligned}\mathrm{Ker}(D^4 - 6D^3 + 13D^2 - 12D + 4) &= \mathrm{Ker}((T-1)^2(T-2)^2) = \mathrm{Ker}(T-1)^2 \oplus \mathrm{Ker}(T-2)^2\\ &= \mathrm{span}(e^x, xe^x) \oplus \mathrm{span}(e^{2x}, xe^{2x})\\ &= \mathrm{span}(e^x, xe^x, e^{2x}, xe^{2x})\end{aligned}$$
which shows that $f(x) = ae^x + bxe^x + ce^{2x} + dxe^{2x}$ for unique $a, b, c, d \in \mathbb{R}$. The given initial conditions $f(0) = f'(0) = f''(0) = f'''(0) = 1$ are satisfied if and only if
$$\begin{aligned}a + b &= 0\\ a + b + 2c + d &= 0\\ a + 2b + 4c + 4d &= 0\\ a + 3b + 8c + 12d &= 0\end{aligned}$$
which has the unique solution $a = 1, b = c = d = 0$. The unique solution to the problem is $f(x) = e^x$.

2. (a) We have $(L-a)(f_0)(n) = (a^{n+1} - a \cdot a^n) = 0$ and, for $i \geq 1$,
$$(L-a)(f_i)(n) = (n+1)^i a^{n+1} - an^i a^n = a((n+1)^i - n^i)a^n = \sum_{m=0}^{i-1} a \binom{i}{m} n^i a^n$$
which shows that $(L-a)(f_1) = aif_0$ and $(L-a)f_i = aif_{i-1} + g$ with $g \in V_{i-1}$. It follows that $(L-a)(V_1) = \{0\}$ and $(L-a)(V_i) \subseteq V_{i-1}$ for $i \geq 2$. Hence $(L-a)^i(V_i) = \{0\}$ for $i \geq 1$. If follows inductively that $(L-a)^i(f_i) = i!a^i f_0$.

   (b) Since $(L-a)^i(V_i) = \{0\}$ for $i \geq 1$ we have $V_k \subseteq \mathrm{Ker}(L-a)^k$.

   (c) If $f_0, f_1, \ldots, f_{k-1}$ are not linearly independent there is a dependence relation $c_0 f_0 + \cdots + c_i f_i = 0$ with $c_i \neq 0$. Applying $(T-a)^i$ to both sides, we get $i!a^i c_i f_0 = 0$ from which $i!a^i c_i = 0$. Since $a^i c_i \neq 0$ we must have $i! = 0$ in F contradicting the fact that F is of characteristic zero. Thus $\dim(V_k) = k$.

   (d) Since $V_k \subseteq \mathrm{Ker}(L-a)^k$ and $\dim(\mathrm{Ker}(L-a)^k) = k$ we have $V_k = \mathrm{Ker}(L-a)^k$.

   (e) We have $g_0 = f_0$ and, for $i \geq 1$,
$$(L-a)(g_i)(n) = \binom{n+1}{i} a^{n+1-i} - a\binom{n}{i} a^{n-i} = \left(\binom{n+1}{i} - \binom{n}{i}\right) a^{n+1-i} = \binom{n}{i-1} a^{n+1-i} = g_{i-1}.$$
Hence $\mathrm{span}(g_0, \ldots, g_{k-1}) \subseteq \mathrm{Ker}(L-a)^k$ and we have equality since $(L-a)^i(g_i) = g_0$ yields the fact that $g_0, \ldots, g_{k-1}$ are linearly independent as above. Consequently, any $f \in \mathrm{Ker}(L-a)^k$ can be uniquely written in the form
$$f = c_0 g_0 + c_1 g_1 + \cdots + c_{k-1} g_{k-1}$$
with $c_0, \ldots c_{k-1} \in F$. Applying, $(L-a)^i$ to both sides and using the fact that $g_(j)(0) = 0$ for $j > 0$, we get
$$(L-a)^i(f)(0) = c_i g_0(0) = c_i.$$

   (f) If $t = (L-1)(s)$ we have $t_n = (n+1)^2 2^{n+1} = 2n^2 2^n + 4n2^n + 2 \cdot 2^n$ so that $t = 2f_0 + 4f_1 + 2f_2 \in \mathrm{Ker}(L-2)^3$. It follows that
$$s \in \mathrm{Ker}(L-1)(L-2)^3 = \mathrm{Ker}(L-1) \oplus \mathrm{Ker}(L-2)^3.$$
Hence $s = cu + c_0 g_0 + c_1 g_1 + c_2 g_2$ where $u = (1, 1, \ldots, 1, \ldots)$ and $g_i(n) = \binom{n}{i}$. Applying $(L-2)^3$ to both sides, we get $(L-1)^3(s) = -cu$. Since $(L-2)^3(s) = (6, \ldots)$ we have $c = -6$. We thus have
$$r = s + 6u = c_0 g_0 + c_1 g_1 + c_2 g_2 = (6, 8, 24, \ldots)$$

so that $c_0 = 6$. Using the fact that $(L-2)(r) = (-4, 8, \ldots)$ and $(L-2)^2(r) = (16, \ldots)$, we get $c_1 = -4, c_2 = 16$. Hence $s_n = -6 + 6 \cdot 2^n - 4 \cdot n2^{n-1} + 16\frac{n(n-1)}{2}2^{n-2} = -6 + 6 \cdot 2^n - n2^{n+2} + n^2 2^{n+1}$.

3. (a) $\Longrightarrow$ (b). If $v_1 + \ldots + v_n = 0$ with $v_i \in V_i$ then $v_i = \sum_{j \neq i} -v_j \in W_i \cap V_i = \{0\} \implies v_i = 0$.

   (b) $\Longrightarrow$ (c). If $v = v_1 + \ldots + v_n = w_1 + \ldots + w_n$ with $v_i, w_i \in V_i$ and $z_i = v_i - w_i \in V_i$ then $z_1 + \ldots + z_n = 0$ which implies $z_i = 0$.

   (c) $\Longrightarrow$ (d). Since B spans V, we only have to show that B is a linearly independent set. Let $c_1 f_1 + \ldots + c_n f_n = 0$ be a dependence relation with $f_1, \ldots, f_n \in B$ and let v_i be the sum of those terms $c_j f_j$ with $f_j \in B_i$. Then $v_1 + \ldots + v_n = 0 = 0 + \ldots + 0$ which implies $v_i = 0$ and hence that $c_j = 0$ if $f_j \in B_i$. Hence $c_j = 0$ for all j since this holds for all i.

   (d) $\Longrightarrow$ (a). Let $v \in V_i \cap W_i$ and let B, B_i be as in (d). If $v = c_1 f_1 + \cdots + c_n f_n$ with $f_1, \ldots, f_n \in B_i$ then

$$c_1 f_1 + \cdots + c_n f_n \in \operatorname{span}(\cup_{j \neq i} B_j)$$

   which implies $c_1 = \cdots c_n = 0$ for otherwise we would get a non-trivial dependence relation among the vectors of B. Hence $v = 0$ and the sum is direct.

4. If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$T(X) = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix},\ T^2(X) = \begin{bmatrix} 2(a-d) & 2(b-c) \\ 2(c-b) & 2(d-a) \end{bmatrix},\ T^3(X) = \begin{bmatrix} 4(c-b) & 4(d-a) \\ 4(a-d) & 4(b-c) \end{bmatrix} = 4T(X)$$

   so that $T^3 = 4T$. Thus $T^3 - 4T = 0$ which gives $T(T-2)(T+2) = 0$. Consequently, the possible eigenvalues of T are $0, \pm 2$. Since

$$\operatorname{Ker}(T) = \operatorname{span}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}), \quad \operatorname{Ker}(T-2) = \operatorname{span}(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}), \quad \operatorname{Ker}(T+2) = \operatorname{span}(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix})$$

   we see that the eigenvalues of T are $0, \pm 2$ and that

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, F_3 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, F_4 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

   is a basis for $\mathbb{R}^{2\times 2}$ consisting of eigenvectors of T. The matrix of T with respect to this basis is

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

   The matrix of T with respect to the basis

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   is the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

   We have $B = P^{-1}AP$, where

$$P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 \end{bmatrix}$$

   is the transition matrix from the E-basis to the F-basis.