McGill University
MATH 251: Algebra 2
Assignment 3 Solutions

1. (a) If $ae^x + be^{2x} + ce^{3x} = 0$ for all x then on differentiating twice we get

$$\begin{aligned} ae^x + be^{2x} + ce^{3x} &= 0 \\ ae^x + 2be^{2x} + 3ce^{3x} &= 0 \\ ae^x + 4be^{2x} + 9ce^{3x} &= 0 \end{aligned}$$

for all x. Setting $x = 0$, we get

$$\begin{aligned} a + b + c &= 0 \\ a + 2b + 3c &= 0 \\ a + 4b + 9c &= 0 \end{aligned}$$

from which $a = b = c = 0$ by Gaussian elimination. Therefore, the given functions are linearly independent.

(b) Using the identity, $\sin(x + a) = \cos(a)\sin(x) + \sin(a)\cos(x)$ we see that the three given functions are in $W = \text{span}(\sin, \cos)$ which is at most 2-dimensional. So the given functions must be linearly dependent since the size of an independent set of vectors in W is ≤ 2.

(c) If $a(x+1)^2 + b(x+2)^2 + c(x+3)^2$ then, applying the same technique of differentiating and setting $x = 0$, we get

$$\begin{aligned} a + 4b + 9c &= 0 \\ 2a + 4b + 6c &= 0 \\ 2a + 2b + 2c &= 0 \end{aligned}$$

from which $a = b = c = 0$ by Gaussian elimination. Hence the given functions are linearly independent.

2. (a) T is linear since $T(aX + bY) = A(aX + bY) - (aX + bY)A = a(AX - XA) + b(AX - XA) = aT(X) + bT(Y)$.

(b) If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $T(X) = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix} = (c-b)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (d-a)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$$

are linearly independent we see that they are a basis for $\text{Im}(T)$. It also follows that $T(X) = 0$ if and only if $b = c$ and $a = d$ so that $X \in \text{Ker}(T)$ if and only iff

$$X = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are linearly independent they are a basis for Ker(T).

(c) Since $T(E_1) = -E_2 + E_3, T(E_2) = -E_1 + E_4, T(E_3) = E_1 - E_4, T(E_4) = E_2 - E_3$ the matrix of T with respect to the basis E_1, E_2, E_3, E_4 is

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

The rank of this matrix is the dimension of its column space which is isomorphic to $\text{Im}(T)$ and hence is 2. The nullity is the dimension of the null-space of A which is isomorphic to the $\ker(T)$ and hence is 2.

3. (a) Let $t = (L-1)(s)$. Then $t_n = s_{n+1} - s_n = (n+1)2^{n+1} = 2n2^n + 2\cdot 2^n$ so that $t = 2v + 2w$, where v, w are defined in (b). We have $(L-2)(v) = 0$ and $(L-2)(w) = 2v$ so that $v, w \in \mathrm{Ker}((L-2)^2)$. Thus $t = (L-1)(s) \in \mathrm{Ker}((L-2)^2)$ which implies $s \in \mathrm{Ker}((L-1)(L-2)^2)$.

(b) Since $u \in \mathrm{Ker}(L-1)$ we have $u, v, w \in W = \mathrm{Ker}((L-1)(L-2)^2)$. Since the dimension of W is 3 the vectors u, v, w will be a basis if we can prove that they are linearly independent. If $au + bv + cw = 0$ then $a + b = 0$, $a + 2b + 2c = 0$, $2b + 8c = 0$ which implies $a = b = c = 0$ by Gaussian elimination.

(c) From (a) and (b) we have $s = au + bv + cw$ for some scalars a, b, c. This yields the system of equations

$$\begin{aligned} a + b &= 0 \\ a + 2b + 2c &= 2 \\ a + 4b + 8c &= 8 \end{aligned}$$

which has the unique solution $a = 2, b = -2, c = 2$. Hence $s_n = 2 - 2^{n+1} + n2^{n+1}$.

4. (a) We have $E_a E_b(f(x)) = e^{ax}e^{bx}f(x) = e^{(a+b)x}f(x) = E_{a+b}(f(x))$ so that $E_aE_b = E_bE_a = I$ if $a + b = 0$ which shows that E_a is invertible with inverse E_{-a}. Note that E_a is linear since $E_a(bf(x) + cg(x)) = e^{ax}(bf(x) + cg(x)) = be^{ax}f(x) + ce^{ax}g(x) = bE_a(f(x)) + cE_a(g(x))$.

(b) We have $(D-a)E_a(f(x)) = (D-a)e^{ax}f(x) = e^{ax}f'(x) = E_aD(f(x))$ so that $(D-a)E_a = E_aD$ which implies that $D - a = E_aDE_a^{-1}$. But then $(D-a)^n = E_aD^nE_a^{-1}$ by induction on n.

(c) From (b) we have

$$(D-a)^n(f(x)) = 0 \iff E_aD^nE_{-a}(f) = 0 \iff D^n(E_{-a}(f)) = 0 \iff E_{-a}(f) \in \mathrm{Ker}(D^n).$$

This shows that $f \in \mathrm{Ker}((D-a)^n) \iff f \in E_a(\mathrm{Ker}(D^n)$. But $1, x, \ldots, x^{n-1}$ is a basis for $\mathrm{Ker}(D^n)$ and E_a is an invertible linear operator on $\mathbb{R}^{\mathbb{R}}$ so that

$$e^{ax}, xe^{ax}, \ldots, x^{n-1}e^{ax}$$

is a basis for $\mathrm{Ker}((D-a)^n)$.