McGill University
MATH 251: Algebra 2
Assignment 2 Solutions

1. Let $S = \{x \in \mathbb{R}^{\mathbb{N}} \mid x_{n+2} = x_{n+1} + 6x_n\}$. Then $S = \mathrm{Ker}(L^2 - L - 6) = \mathrm{Ker}((L-3)(L+2)) = \mathrm{Ker}(L-3) + \mathrm{Ker}(L+2) = \mathrm{span}((1,-2,\ldots,(-2)^n,\ldots)) + \mathrm{span}((1,3,\ldots,3^n,\ldots)) = \mathrm{span}(u,v)$ where $u = (1,-2,\ldots,(-2)^n,\ldots)$, $v = (1,3,\ldots,3^n,\ldots)$ so that $S = \{au+bv \mid a,b \in \mathbb{R}\}$. If $x \in S$ then there are $a,b \in \mathbb{R}$ with $x_n = a(-2)^n + b3^n$ for $n \geq 0$. If $x_0 = x_1 = 1$, we have $a+b = 1$, $-2a+3b = 1$ so that $a = -1/5$, $b = 1/5$. Hence $x_n = (3^n - (-2)^n)/5$.

   Remark. Since $x_n \in \mathbb{Z}$, it follows that $3^n - (-2)^n$ is divisible by 5 for all n. This also follows from the fact that -2 is congruent to 3 mod 5.

2. We first note that $f'' = f' + 6f$ implies $f^{(n)} = f^{(n-1)} - f^{(n-2)}$ for all $n \geq 2$ so that f is infinitely differentiable. Hence, if V is the vector space of infinitely differentiable real-valued functions on the the real line, $S = \{f \in \mathbb{R}^{\mathbb{R}} | f'' = f' + 6f\} = \{f \in V | f'' = f' + 6f\}$. If D is the differentiation operator on V then $S = \mathrm{Ker}(D^2 - D - 6) = \mathrm{Ker}(D-3)(D+2) = \mathrm{Ker}(D-3) + (D+2) = \mathrm{span}(e^{3x}) + \mathrm{span}(e^{-2x}) = \mathrm{span}(e^{3x}, e^{-2x}) = \{ae^{3x} + be^{-2x} \mid a,b \in \mathbb{R}\}$. If $f \in S$ there are $a,b \in \mathbb{R}$ with $f(x) = ae^{3x} + be^{-2x}$ for all x. If $f(0) = f'(0) = 1$, we have $a+b = 1$, $3a - 2b = 1$ so that $a = 1/5$, $b = -1/5$. Hence $f(x) = (e^{3x} - e^{-2x})/5$.

3. We use the fact that the left shift operator on $S = \mathrm{Ker}(L^2 - L - 6)$ corresponds to left multiplication on $\mathbb{R}^{2\times 1}$ by $A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$ using the isomorphism $\phi : S \to \mathbb{R}^{2\times 1}$ defined by $\phi(x) = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. Since $L(u) = -2u$ and $L(v) = 3v$ where $u = (1,2,\ldots,(-2)^n,\ldots)$, $v = (1,3,\ldots,3^n,\ldots)$, we have

$$A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

   Now $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = (\frac{3}{5}x_0 - \frac{1}{5}x_1)\begin{bmatrix} 1 \\ -2 \end{bmatrix} + (\frac{2}{5}x_0 + \frac{1}{5}x_1)\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ which implies that

$$\begin{aligned} A^n\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= (\frac{3}{5}x_0 - \frac{1}{5}x_1)A^n\begin{bmatrix} 1 \\ -2 \end{bmatrix} + (\frac{2}{5}x_0 + \frac{1}{5}x_1)A^n\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= (-2)^n(\frac{3}{5}x_0 - \frac{1}{5}x_1)\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 3^n(\frac{2}{5}x_0 + \frac{1}{5}x_1)\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3(-2)^n + 2\cdot 3^n}{5}x_0 + \frac{3^n - (-2)^n}{5}x_1 \\ \frac{3(-2)^{n+1} + 2\cdot 3^{n+1}}{5}x_0 + \frac{3^{n+1} - (-2)^{n+1}}{5}x_1 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 3(-2)^n + 2\cdot 3^n & 3^n - (-2)^n \\ 3(-2)^{n+1} + 2\cdot 3^{n+1} & 3^{n+1} - (-2)^{n+1} \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \end{aligned}$$

   so that

$$A^n = \frac{1}{5}\begin{bmatrix} 3(-2)^n + 2\cdot 3^n & 3^n - (-2)^n \\ 3(-2)^{n+1} + 2\cdot 3^{n+1} & 3^{n+1} - (-2)^{n+1} \end{bmatrix}.$$

4. To find $U \cap V$, we have to find those vectors $y_1(1,1,1,0,1) + y_2(0,1,1,1,0) + y_3(1,1,1,1,1)$ of V that lie in U. This is equivalent to the finding of x_1, x_2, x_3, x_4 such that

$$x_1(1,1,0,1,0) + x_2(0,1,1,0,1) + x_3(1,0,1,0,1) + x_4(1,1,0,1,1) = y_1(1,1,1,0,1) + y_2(0,1,1,1,0) + y_3(1,1,1,1,1).$$

   This vector equation is equivalent to the system of equations

$$\begin{aligned} x_1 + x_3 + x_4 &= y_1 + y_3 \\ x_1 + x_2 + x_4 &= y_1 + y_2 + y_3 \\ x_2 + x_3 &= y_1 + y_2 + y_3 \\ x_1 + x_4 &= y_2 + y_3 \\ x_2 + x_3 + x_4 &= y_1 + y_3 \end{aligned}$$

Applying Gaussian elimination, we get the equivalent system

$$\begin{aligned}
x_1 + x_3 + x_4 &= y_1 + y_3 \\
x_2 + x_3 &= y_2 \\
x_3 &= y_1 + y_2 \\
x_4 &= y_1 + y_2 + y_3 \\
0 &= y_1 + y_3
\end{aligned}$$

which has a solution if and only if $y_1 = y_3$. Hence

$$\begin{aligned}
U \cap V &= \{y_1(1,1,1,0,1) + y_2(0,1,1,1,0) + y_1(1,1,1,1,1) \mid y_1, y_2 \in \mathbb{F}_2\} \\
&= \{y_1(0,0,0,1,0) + y_2(0,1,1,1,0) \mid y_1, y_2 \in \mathbb{F}_2\} \\
&= \operatorname{span}((0,0,0,1,0),(0,1,1,1,0))
\end{aligned}$$

To find a basis for
$U + V = \operatorname{span}((1,1,0,1,0),(0,1,1,0,1),(1,0,1,0,1),(1,1,0,1,1),(1,1,1,0,1),(0,1,1,1,0),(1,1,1,1,1))$
we find the dependence relations for the given generating set. We have

$$x_1(1,1,0,1,0)+x_2(0,1,1,0,1)+x_3(1,0,1,0,1)+x_4(1,1,0,1,1)+x_5(1,1,1,0,1)+x_6(0,1,1,1,0)+x_7(1,1,1,1,1) = 0$$

if and only if $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ is a solution of the system

$$\begin{aligned}
x_1 + x_3 + x_4 + x_5 + x_7 &= 0 \\
x_1 + x_2 + x_4 + x_5 + x_6 + x_7 &= 0 \\
x_2 + x_3 + x_5 + x_6 + x_7 &= 0 \\
x_1 + x_4 + x_6 + x_7 &= 0 \\
x_2 + x_3 + x_4 + x_5 + x_7 &= 0
\end{aligned}$$

which, by Gaussian elimination is equivalent to the system

$$\begin{aligned}
x_1 &= x_7 \\
x_2 &= x_7 \\
x_3 &= x_6 + x_7 \\
x_4 &= x_6 \\
x_5 &= x_7.
\end{aligned}$$

Since there is a solution with $x_6 = 1, x_7 = 0$ and one with $x_6 = 0, x_7 = 1$, the first five vectors span $U + V$. Moreover, these vectors are linearly independent since the only solution of the above system with $x_6 = x_7 = 0$ is the zero solution.

5. The vectors $u_1 = (1,1,0,1), u_2 = (0,1,1,1), u_3 = (0,0,1,0), u_4 = (0,0,0,1)$ form a basis for $\mathbb{R}^4$ since the equation $(x_1, x_2, x_3, x_4) = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4$ has the unique solution

$$a_1 = x_1, a_2 = x_2 - x_1, a_3 = x_1 - x_2 + x_3, a_4 = x_4 - x_2.$$

If T is the linear mapping with $T(u_1) = T(u_2) = 0, T(u_3) = (1,1,0,1), T(u_4) = (0,1,1,1)$, we have

$$T(x_1, x_2, x_3, x_4) = a_3(1,1,0,1) + a_4(0,1,1,1) = (x_1 - x_2 + x_3, x_1 - 2x_2 + x_3 + x_4, x_4 - x_2, x_1 - 2x_2 + x_3 + x_4)$$

which has the required kernel and image.