McGill University
MATH 251: Algebra 2
Assignment 1 Solutions

1. If V' is a vector space and f is an isomorphism then $f(u+v) = f(u) + f(v), f(cv) = cf(v)$. Setting $u' = f(u), v' = f(v)$, we see that

$$u' + v' = f(f^{-1}(u') + f^{-1}(v')), \quad cv' = f(cf^{-1}(v'))$$

so that the operations of addition and multiplication by scalars on V' are completely determined by f and those of V. Conversely, if we define addition and multiplication on V' by the above formulae, we have $f(u+v) = f(u) + f(v), f(cu) = cf(v)$. Verifying the vector spaces axioms for V' with these operations, we have on setting $u' = f(u), v' = f(v), w' = g(w)$

V1: $(u'+v')+w' = (f(u)+f(v))+f(w) = f((u+v)+w) = f(u+(v+w)) = f(u)+(f(v)+f(w)) = u' + (v' + w')$

V2: $u' + v' = f(u) + f(v) = f(u+v) = f(v+u) = f(v) + f(u) = v' + u'$

V3: Setting $0' = f(0)$, we have $0' + v' = f(0) + f(v) = f(0+v) = f(v) = v'$

V4: Setting $-v' = f(-v)$, we have $-v' + v = f(-v) + f(v) = f(-v+v) = f(0) = 0'$

V5: $1v' = 1f(v) = f(1v) = f(v) = v'$

V6: $a(bv') = a(bf(v)) = af(bv) = f(a(bv)) = f((ab)v) = (ab)f(v) = (ab)v'$

V7: $(a+b)v' = (a+b)f(v) = f((a+b)v) = f(av+bv) = f(av) + f(bv) = af(v) + bf(v) = av' + bv'$, $a(u'+v') = a(f(u)+f(v)) = a(f(u+v)) = f(a(u+v)) = f(au+av) = f(au) + f(av) = af(u) + af(v) = au' + bv'$

It follows that, with these operations, V' is a vector space over F and f is an isomorphism.

2. If $A = [a_{ij}] \in F^{p\times p}$, define $f(A) = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \in \prod_{1\le i,j\le 2} V_{ij}$, where $B = [b_{ij}] \in V_{11}, C = [c_{ij}] \in V_{12}, D = [d_{ij}] \in V_{21}, E = [e_{ij}] \in V_{22}$, by

$$b_{ij} = a_{ij}, \quad c_{ij} = a_{ij-n}, \quad d_{ij} = a_{i-nj}, \quad e_{ij} = a_{i-nj-n}.$$

Conversely, if $X = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \in \prod_{1\le i,j\le 2} V_{ij}$, define $A = g(X)$ by

$$a_{ij} = \begin{cases} b_{ij} & \text{if } 1 \le i,j \le n, \\ c_{ij} & \text{if } 1 \le i \le n, n+1 \le j \le n+m, \\ d_{ij} & \text{if } n+1 \le i \le n+m, 1 \le j \le n, \\ d_{ij} & \text{if } n+1 \le i \le n+m, n+1 \le j \le n+m. \end{cases}$$

Then f and g are inverse functions which proves that f is bijective. If $f(A') = \begin{bmatrix} B' & C' \\ D' & E' \end{bmatrix}$, we have

$$f(sA + tA') = \begin{bmatrix} sB + tB' & sC + tC' \\ sD + tD' & sE + tE' \end{bmatrix} = s\begin{bmatrix} B & C \\ D & E \end{bmatrix} + t\begin{bmatrix} B' & C' \\ D' & E' \end{bmatrix} = sf(A) + tf(B),$$

which shows that $f : F^{p\times p} \to \prod_{1\le i,j\le 2} V_{ij}$ is an isomorphism of vector spaces.

3. Suppose that the vector space W is the union of the two proper subspaces U, V. Then $U \subsetneq V$; otherwise $V = W$. Similarly, $V \subsetneq U$. Let $v \in V, v \notin U$ and $U \in U, u \notin V$. Then $w = u + v$ must be in U or V. If $w \in U$ then $v = w + (-u) \in U$ which is a contradiction. If $w \in V$ then $u = w + (-v) \in V$ which is also a contradiction. Hence, the original hypothesis ($W = U \cup V$) is incorrect.

   Note that the above argument yields the fact that the union of two proper subspaces, neither of which is contained in the other, is not a subspace.

   If $V = \mathbb{F}_2$ and $U_1 = \{(0,0),(1,0)\}$, $U_2 = \{(0,0),(0,1)\}$, $U_3 = \{(0,0),(1,1)\}$ then U_1, U_2, U_3 are proper subspaces of V and $V = U_1 \cup U_2 \cup \cup U_3$.

4. (a) Since $x^2 + xy + y^2 = (x + y/2)^2 + 3y^2/4$, we have $x^2 + xy + y^2 = 0 \iff x = y = 0$. Therefore $S = \{(0,0)\}$ which is a subspace of $\mathbb{R}^2$, the zero subspace.

   (b) Since $x^2 + 2xy + y^2 = (x+y)^2$, we have $S = \{(x,y) \mid x + y = 0\}$ which is a subspace of $\mathbb{R}^2$.

   (c) Since $x^2 + 2xy - 8y^2 = (x+4y)(x-2y)$, we see that $u = (4,-1), v = (2,1) \in S$ but $u+v = (6,0) \notin S$. Therefore S is not a subspace of $\mathbb{R}^2$.

   (d) If x is the zero sequence then $x_n = 0$ for all n which implies that $x_n^2 = nx_n$ for all $n \geq 0$. Thus the zero vector is in S. If $x, y \in S$ and $a, b \in \mathbb{R}$ then $(ax+by)_n^2 = ax_n^2 + by_n^2 = anx_n + bny_n = n(xax+by)_n$ which shows that $ax + by \in S$. Thus S is a subspace of $\mathbb{R}^{\mathbb{N}}$.

   (e) If f is the zero function then $f(x) = 0$ for all $x \in \mathbb{R}$ so that $f(x+2) - f(x+1) = xf(x) = 0$ for all $x \in \mathbb{R}$. If $f, g \in S$ and $a, b \in \mathbb{R}$, we have

$$\begin{aligned}(af+bg)(x+2) - (af+bg)(x+1) &= af(x+2) + bg(x+2) - af(x+1) - bg(x+1) \\ &= a(f(x+2) - f(x+1)) + b(g(x+2) - g(x+1)) \\ &= a(xf(x)) + b(xg(x) = x(af+bg)(x)\end{aligned}$$

   which shows that $af + bg \in S$. Hence S is a subspace of $\mathbb{R}^{\mathbb{R}}$.

5. If x is the zero sequence then $x_{n+k} = a_1x_{n+k-1} + a_2x_{n+k-2} + \cdots a_kx_n = 0$ for all $n \geq 0$. This implies that S contains the zero vector. If $x, y \in S$ and $a, b \in F$ then

$$\begin{aligned}(ax+by)_{n+k} &= ax_{n+k} + by_{n+k} \\ &= a(a_1x_{n+k-1} + a_2x_{n+k-2} + \cdots + a_kx_n) + b(a_1y_{n+k-1} + a_2y_{n+k-2} + \cdots + a_kx_n + a_ky_n \\ &= aa_1x_{n+k-1} + ba_1y_{n+k-1} + aa_2x_{n+k-2} + ba_2y_{n+k-2} + \cdots + aa_kx_n + ba_ky_n \\ &= a_1((ax+by)_{n+k-1} + a_2(ax+by)_{n+k-2} + \cdots + a_k(ax+by)_n\end{aligned}$$

   which shows that $ax + by \in S$ and hence that S is a subspace of $F^{\mathbb{N}}$. Since

$$\begin{aligned}T(ax+by) &= (ax_0 + by_0, ax_1 + by_1, \ldots, ax_{k-1} + by_{k-1}) \\ &= a(x_0, x_1, \ldots, x_{k-1}) + b(y_0, y_1, \ldots, y_{k-1}) = aT(x) + bT(y)\end{aligned}$$

   we see that T is a linear mapping. Now define a mapping $R : F^k \to S$ by defining $R(x_0, x_1, \ldots, x_{k-1}) = (x_0, x_1, \ldots, x_n, \ldots)$ where x_n is defined inductively for $n \geq k$ by

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \ldots + a_kx_{n-k}.$$

   Then R and T are inverse functions which shows that T is bijective and hence that T is an isomorphism.