McGill University
MATH 251: Algebra 2
Assignment 1 Solutions

1. If V' is a vector space and f is an isomorphism then $f(u+v)=f(u)+f(v), f(cv)=cf(v)$. Setting $u'=f(u), v'=f(v)$, we see that

$$u'+v'=f(f^{-1}(u')+f^{-1}(v')), \quad cv'=f(cf^{-1}(v'))$$

so that the operations of addition and multiplication by scalars on V' are completely determined by f and those of V. Conversely, if we define addition and multiplication on V' by the above formulae, we have $f(u+v)=f(u)+f(v), f(cu)=cf(v)$. Verifying the vector spaces axioms for V' with these operations, we have on setting $u'=f(u), v'=f(v), w'=g(w)$

V1: $(u'+v')+w'=(f(u)+f(v))+f(w)=f((u+v)+w)=f(u+(v+w))=f(u)+(f(v)+f(w))=u'+(v'+w')$

V2: $u'+v'=f(u)+f(v)=f(u+v)=f(v+u)=f(v)+f(u)=v'+u'$

V3: Setting $0'=f(0)$, we have $0'+v'=f(0)+f(v)=f(0+v)=f(v)=v'$

V4: Setting $-v'=f(-v)$, we have $-v'+v=f(-v)+f(v)=f(-v+v)=f(0)=0'$

V5: $1v'=1f(v)=f(1v)=f(v)=v'$

V6: $a(bv')=a(bf(v))=af(bv)=f(a(bv))=f((ab)v)=(ab)f(v)=(ab)v'$

V7: $(a+b)v'=(a+b)f(v)=f((a+b)v)=f(av+bv)=f(av)+f(bv)=af(v)+bf(v)=av'+bv'$, $a(u'+v')=a(f(u)+f(v))=a(f(u+v))=f(a(u+v))=f(au+av)=f(au)+f(av)=af(u)+af(v)=au'+bv'$

It follows that, with these operations, V' is a vector space over F and f is an isomorphism.

2. If $A=[a_{ij}]\in F^{p\times p}$, define $f(A)=\begin{bmatrix} B & C \\ D & E \end{bmatrix}\in\prod_{1\le i,j\le 2}V_{ij}$, where $B=[b_{ij}]\in V_{11}, C=[c_{ij}]\in V_{12}, D=[d_{ij}]\in V_{21}, E=[e_{ij}]\in V_{22}$, by

$$b_{ij}=a_{ij}, \quad c_{ij}=a_{ij-n}, \quad d_{ij}=a_{i-nj}, \quad e_{ij}=a_{i-nj-n}.$$

Conversely, if $X=\begin{bmatrix} B & C \\ D & E \end{bmatrix}\in\prod_{1\le i,j\le 2}V_{ij}$, define $A=g(X)$ by

$$a_{ij}=\begin{cases} b_{ij} & \text{if } 1\le i,j\le n, \\ c_{ij} & \text{if } 1\le i\le n, n+1\le j\le n+m, \\ d_{ij} & \text{if } n+1\le i\le n+m, 1\le j\le n, \\ d_{ij} & \text{if } n+1\le i\le n+m, n+1\le j\le n+m. \end{cases}$$

Then f and g are inverse functions which proves that f is bijective. If $f(A')=\begin{bmatrix} B' & C' \\ D' & E' \end{bmatrix}$, we have

$$f(sA+tA')=\begin{bmatrix} sB+tB' & sC+tC' \\ sD+tD' & sE+tE' \end{bmatrix}=s\begin{bmatrix} B & C \\ D & E \end{bmatrix}+t\begin{bmatrix} B' & C' \\ D' & E' \end{bmatrix}=sf(A)+tf(B),$$

which shows that $f: F^{p\times p}\to\prod_{1\le i,j\le 2}V_{ij}$ is an isomorphism of vector spaces.

3. Suppose that the vector space W is the union of the two proper subspaces U, V. Then $U \subsetneq V$; otherwise $V = W$. Similarly, $V \subsetneq U$. Let $v \in V, v \notin U$ and $U \in U, u \notin V$. Then $w = u + v$ must be in U or V. If $w \in U$ then $v = w + (-u) \in U$ which is a contradiction. If $w \in V$ then $u = w + (-v) \in V$ which is also a contradiction. Hence, the original hypothesis ($W = U \cup V$) is incorrect.

   Note that the above argument yields the fact that the union of two proper subspaces, neither of which is contained in the other, is not a subspace.

   If $V = \mathbb{F}_2$ and $U_1 = \{(0,0),(1,0)\}$, $U_2 = \{(0,0),(0,1)\}$, $U_3 = \{(0,0),(1,1)\}$ then U_1, U_2, U_3 are proper subspaces of V and $V = U_1 \cup U_2 \cup \cup U_3$.

4. (a) Since $x^2 + xy + y^2 = (x + y/2)^2 + 3y^2/4$, we have $x^2 + xy + y^2 = 0 \iff x = y = 0$. Therefore $S = \{(0,0)\}$ which is a subspace of $\mathbb{R}^2$, the zero subspace.

   (b) Since $x^2 + 2xy + y^2 = (x + y)^2$, we have $S = \{(x, y) \mid x + y = 0\}$ which is a subspace of $\mathbb{R}^2$.

   (c) Since $x^2+2xy-8y^2 = (x+4y)(x-2y)$, we see that $u = (4,-1), v = (2,1) \in S$ but $u+v = (6,0) \notin S$. Therefore S is not a subspace of $\mathbb{R}^2$.

   (d) If x is the zero sequence then $x_n = 0$ for all n which implies that $x_n^2 = nx_n$ for all $n \geq 0$. Thus the zero vector is in S. If $x, y \in S$ and $a, b \in \mathbb{R}$ then $(ax + by)_n^2 = ax_n^2 + by_n^2 = anx_n + bny_n = n(xax + by)_n$ which shows that $ax + by \in S$. Thus S is a subspace of $\mathbb{R}^{\mathbb{N}}$.

   (e) If f is the zero function then $f(x) = 0$ for all $x \in \mathbb{R}$ so that $f(x+2) - f(x+1) = xf(x) = 0$ for all $x \in \mathbb{R}$. If $f, g \in S$ and $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
(af + bg)(x+2) - (af + bg)(x+1) &= af(x+2) + bg(x+2) - af(x+1) - bg(x+1) \\
&= a(f(x+2) - f(x+1)) + b(g(x+2) - g(x+1)) \\
&= a(xf(x)) + b(xg(x) = x(af + bg)(x)
\end{aligned}$$

   which shows that $af + bg \in S$. Hence S is a subspace of $\mathbb{R}^{\mathbb{R}}$.

5. If x is the zero sequence then $x_{n+k} = a_1 x_{n+k-1} + a_2 x_{n+k-2} + \cdots a_k x_n = 0$ for all $n \geq 0$. This implies that S contains the zero vector. If $x, y \in S$ and $a, b \in F$ then

$$\begin{aligned}
(ax + by)_{n+k} &= ax_{n+k} + by_{n+k} \\
&= a(a_1 x_{n+k-1} + a_2 x_{n+k-2} + \cdots + a_k x_n) + b(a_1 y_{n+k-1} + a_2 y_{n+k-2} + \cdots + a_k x_n + a_k y_n \\
&= aa_1 x_{n+k-1} + ba_1 y_{n+k-1} + aa_2 x_{n+k-2} + ba_2 y_{n+k-2} + \cdots + aa_k x_n + ba_k y_n \\
&= a_1((ax + by)_{n+k-1} + a_2(ax + by)_{n+k-2} + \cdots + a_k(ax + by)_n
\end{aligned}$$

   which shows that $ax + by \in S$ and hence that S is a subspace of $F^{\mathbb{N}}$. Since

$$\begin{aligned}
T(ax + by) &= (ax_0 + by_0, ax_1 + by_1, \ldots, ax_{k-1} + by_{k-1}) \\
&= a(x_0, x_1, \ldots, x_{k-1}) + b(y_0, y_1, \ldots, y_{k-1}) = aT(x) + bT(y)
\end{aligned}$$

   we see that T is a linear mapping. Now define a mapping $R : F^k \to S$ by defining $R(x_0, x_1, \ldots, x_{k-1}) = (x_0, x_1, \ldots, x_n, \ldots)$ where x_n is defined inductively for $n \geq k$ by

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_k x_{n-k}.$$

   Then R and T are inverse functions which shows that T is bijective and hence that T is an isomorphism.