Notes on the Dual Space

Let V be a vector space over a field F. The **dual space** of V is the vector space $V^* = \mathrm{Lin}(V, F)$ consisting of the linear mappings $\phi : V \to F$. The elements of V^* are called linear forms or linear functionals. If V has basis $e = (e_i)_{i\in I}$ then any vector $v \in V$ can be written uniquely in the form $v = \sum_{i\in I} x_i e_i$ with $x_i = 0$ for all but a finite number of $i \in I$. If $\phi \in V^*$ then $\phi(v) = \sum_{i\in I} a_i x_i$ where $a_i = f(e_i)$. Conversely, if $(a_i)_{i\in I}$ is a family of scalars indexed by I then

$$\phi(v) = \sum_{i\in I} a_i x_i$$

defines a linear form ϕ on V with $\phi(e_i) = a_i$. In particular, $e_i^*(v) = x_i$ defines a linear form on V called the i-th coordinate function for the basis $e = (e_i)_{i\in I}$. If I is a finite set then any $\phi \in V^*$ can be uniquely written in the form

$$\phi = \sum_{i\in I} a_i e_i^*$$

with $a_i \in F$. We have $a_i = \phi(e_i)$ since $e_i^*(e_j) = \delta_{ij}$ (the Kronecker delta). Hence, in this case, $e^* = (e_i^*)_{i\in I}$ is a basis for V^*, the basis of V^* dual to the basis $e = (e_i)_{i\in I}$ or **dual basis**. We thus have the result

Theorem 1. *If V is a finite-dimensional vector space then* $\dim(V) = \dim(V^*)$.

If the basis (e_i) is infinite then the subspace W of V^* spanned by the vectors $e_i^*(i \in I)$ is a proper subspace of V. For example, the linear form ϕ with $\phi(e_i) = 1$ for all i is not in W since $\phi(e_i) = 0$ for all but a finite number of i if $\phi \in W$. In fact, using set theory, one can show that $\dim(V^*) > \dim(V)$ if $\dim(V)$ is not finite.

Example 1. *If $V = F^X$ is the set of F-valued functions on X then, for any $x \in X$, the function $e_x :\to F$ defined by $e_x(f) = f(x)$ (evaluation at x) is a linear form on V.*

Example 2. *If $V = C_{\mathbb{R}}([a, b])$ is the vector space of continuous real-valued functions on the interval $[a, b]$ then*

$$\phi(f) = \int_a^b f(x)\, dx$$

defines a linear form on V.

Example 3. *If $V = F^n$ and $\phi \in V^*$ then*

$$\phi(x_1, \dots, x_n) = a_1x_1 + \cdots a_nx_n = [a_1, a_2, \dots, a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $[a_1, a_2, \dots, a_n] = [\phi]$ is the matrix of ϕ.

If S is a subset of the vector space V then the **annihilator** of S is the set

$$S^0 = \{\phi \in V^* \mid \phi(s) = 0 \text{ for all } s \in S\}$$

consisting of those linear forms on V which vanish on S. We have

$$S^0 = (\mathrm{span}(S))^0.$$

If W is a subspace of F^n then the elements of W^0 give the hyperplanes $a_1x_1 + \cdots + a_nx_n = 0$ which contain W.

Example 4. *If* $W = \mathrm{Span}((1,1,1))$ *then* $W^0 = \mathrm{Span}(\phi_1, \phi_2)$ *where* $\phi_1(x_1,x_2,x_3) = x_1 - x_2$*,* $\phi_2(x_1,x_2,x_3) = x_2 - x_3$*.*

If W is a subspace of V we can define a linear mapping $T : W^0 \to (V/W)^*$ by

$$T(\phi)(v + W) = \phi(v).$$

The function $T(\phi)$ is well defined on V/W since ϕ vanishes on W. That $T(\phi)$ is linear is left to the reader as well as the fact that T is injective. If ψ is a linear form on V/W then $\phi(v) = \psi(v + W)$ defines a linear form on V and $\psi = T(\phi)$ which shows that T is surjective and hence an isomorphism.

Theorem 2. *If* W *is a subspace of the vector space* V *then*

$$W^0 \cong (V/W)^*.$$

If $\dim(V/W) < \infty$ *then*

$$\dim(W^0) = \mathrm{codim}(W),$$

where $\mathrm{codim}(W) = \dim(V/W)$*. If* $\dim(V) < \infty$*, we have*

$$\dim(W^0) = \dim(V) - \dim(W).$$

There is a canonical mapping R of a vector space V into its second dual $V^{**} = (V^*)^*$ defined by $R(v) = v^{**}$ where $v^{**}(\phi) = \phi(v)$. The proof of the linearity of v^{**} and R are left to the reader. If $R(v) = 0$ we have $\phi(v) = 0$ for all $\phi \in V^*$. If $v \neq 0$ then it can be completed to a basis B of V. Then the linear form v^* which is 1 on v and 0 on the other vectors in B contradicts the fact that $\phi(v) = 0$ for all $\phi \in V^*$. If V is finite-dimensional, the mapping R is surjective since $\dim(V) = \dim(V^{**})$. In this case, we use the isomorphism R to identify V with its second dual.

If W is a subspace of V^*, we define the annihilator of W in V to be the set

$$W^0 = \{v \in V : \phi(v) = 0 \text{ for all } \phi \in W\}.$$

If $\dim(V) < \infty$ then $R(W^0) = \{v^{**} | v^{**}(\phi) = 0 \text{ for all } \phi \in W\}$ is the annihilator of W in V^{**} which gives the following result.

Theorem 3. *If* V *is finite-dimensional and* W *is a subspace of* V^* *then* $\dim(W^0) = \mathrm{codim(W)}$*.*

This can also be proven by showing that the mapping $v \mapsto v_*$ of W^0 into $(V^*/W)^*$, where $v_*(\phi + W) = \phi(v)$ is an isomorphism when V is finite-dimensional.

Corollary 4. *If* $\dim(V) < \infty$ *and* W *is a subspace of* V *or* V^* *then* $W^{00} = W$*.*

This follows from the fact that $W \subseteq W^{00}$. Since $W_1 \subseteq W_2$ implies $W_2^0 \subseteq W_1^0$ we get, in the case V is finite-dimensional, an inclusion reversing bijection $W \mapsto W^0$ between the the subspaces of V and the subspaces of W. The corresponding subspaces are said to be dual to one another.

Theorem 5. *If* W_1, W_2 *are subspaces of a vector space* V *we have* $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$*. If* $\dim(V) < \infty$ *we also have* $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$*.*

The proof is left to the reader.

If $T : U \to V$ is a linear mapping the transpose of T is the mapping $T^t : V^* \to U^*$ defined by

$$T^t(\phi) = \phi \circ T.$$

The proof that T^t is linear is left to the reader as is the fact that the mapping $T \mapsto T^t$ is linear. If U, V are finite-dimensional then $(T^t)^t(u^{**}) = u^{**} \circ T^t$ and

$$u^{**} \circ T^t(\phi) = u^{**} \circ \phi \circ T = \phi(T(u) = T(u)^{**}(\phi)$$

for $\phi \in V^{**}$ so that $(T^t)^t = T$ after identifying U and V with their duals. If $S : V \to W$ then $(ST)^t = T^t S^t$ since

$$(ST)^t(\phi) = \phi \circ ST = S^t(\phi) \circ T = T^t(S^t(\phi)) = T^t S^t(\phi)$$

for $\phi \in U^*$.

Theorem 6. *If $T : U \to V$ is linear then*

$$(\mathrm{Im(T)})^0 = \mathrm{Ker(T^t)} \textit{ and } (\mathrm{Im(T^t)})^0 = \mathrm{Ker(T)}.$$

The proof is left to the reader.

If U, V are vector spaces with bases $e = (e_1, \dots, e_)$, $f = (f_1, \dots, f_m)$ respectively and $T : U \to V$ is a linear mapping then

$$T(e_j) = \sum_{i=1}^{m} a_{ij} f_i.$$

The $m \times n$ matrix $A = [a_{ij}]$ is the matrix $[T]_e^f$ of T with respect to the bases e, f. If $e^* = (e_1^*, \dots, e_n^*)$, $f^* = (f_1^*, \dots, f_m^*)$ are the corresponding dual bases then

$$T^t(f_i^*) = \sum_{j=1}^{n} b_{ji} e_j^*.$$

The matrix $n \times m$ matrix $B = [b_{ji}]$ is the matrix $[T^t]_{f^*}^{e^*}$ of T^t with respect to the bases f^*, e^*. Since

$$b_{ji} = T^t(f_i^*)(e_j) = f_i^*(T(e_j)) = a_{ij},$$

we see the the (j, i)-th entry of B is the (i, j)-th entry of A. The matrix B is called the transpose of A and is denoted by A^t.

Theorem 7. *If U, V are vector spaces with bases $e = (e_1, \dots, e_)$, $f = (f_1, \dots, f_m)$ respectively and $T : U \to V$ is a linear mapping then*

$$[T^t]_{f^*}^{e^*} = ([T]_e^f)^t.$$

The linearity of the transpose for matrices as well as $(A^t)^t = A$ and $(AB)^t = B^t A^t$ follow from the above.

If $A \in F^{m \times n}$, we have two linear mappings $T_A : F^{n \times 1} \to F^{m \times 1}$, $T_A : F^{1 \times m} \to F^{1 \times n}$ defined by

$$T_A(X) = AX, \quad T^A(Y) = AY.$$

If we identify $(F^{n \times 1})^*$ with $F^{1 \times n}$ using the isomorphism $\phi \mapsto [\phi]$ then $(T_A)^t = T^A$. We have

$$\mathrm{Ker(T_A) = Null(A), \quad Im(T_A) = Col(A), \quad Ker(T^A) \cong Null(A^t), \quad Im(T^A) = Row(A) \cong Col(A^t)}$$

where the isomorphisms are defined by the transpose. If W is a subset of $F^{n \times 1}$ then

$$W^0 = \{Y \in F^{1 \times n} \mid YX = 0 \text{ for all } X \in W\}$$

and if W is a subset of $F^{1 \times n}$ then

$$W^0 = \{X \in F^{n \times 1} \mid YX = 0 \text{ for all } Y \in W\}.$$

Since $\mathrm{(Im(T_A)^0 = Null(T^A)}$ and $\mathrm{(Im(T^A))^0 = Ker(T_A)}$, we have

$$\mathrm{Col(A)^0 = \{Y \in F^{1 \times m} \mid YA = 0\} = \{Y \in F^{1 \times m} \mid A^t Y^t = 0\}, \quad Row(A)^0 = Null(A)}$$

It follows that

$$\mathrm{rank(A) + nullity(A^t) = m, \quad rank(A^t) + nullity(A) = n.}$$

In particular, $\mathrm{rank(A) = m - nullity(A^t) = rank(A^t)}$. Since $\mathrm{Row(A) = Row(A)^{00}}$, we also have

$$\mathrm{Row(A) = Null(A)^0}.$$

Thus two $m \times n$ matrices which have the same also have the same row space.

There is a canonical form for the row space of a matrix A, namely, its **reduced echelon form** $\mathrm{REF}(A)$. This $m \times n$ matrix is row equivalent to A and so has the same row space as A. It has the following properties:

(a) The non-zero rows precede the zero rows;

(b) The first non-zero entry in a non-zero row is 1;

(c) If the first-nonzero entry in row i occurs in column j_i then $j_1 < j_2 < \cdots < j_s$, where s is the number of non-zero rows;

(d) For $1 \leq i \leq s$, there is exactly one non-zero entry in column j_i.

The natural number s is the dimension of the row space of A.

Theorem 8. *Let A, A' be $m \times n$ matrices over a field F. Then the following are equivalent:*

(1) *There an invertible matrix P such that $A' = PA$;*

(2) $\mathrm{Row(A)} = \mathrm{Row(A')}$*;*

(3) $\mathrm{REF}(A) = \mathrm{REF}(A')$.

Proof. As (1) $\implies$ (2) and (3) $\implies$ (1) are immediate, we only give the proof of (2) $\implies$ (3). We proceed by induction on the common number of non-zero rows of A and B. Let $A_1, \ldots, A_s$ be the non-zero rows of A and let $A_1', \ldots, A_s'$ be the non-zero rows of A'. Let $j_1, \ldots, j_s$ and $j_1', \ldots, j_s'$ be the distinguished columns of A and B respectively. Since $A_1 = A_1' + a_2 A_2' + \ldots + A_s'$ we have $j_1 = j_1'$. It follows that

$$\mathrm{Span(A_2, \ldots, A_s)} = \mathrm{Span(A_2', \ldots, A_s')}.$$

Thus the matrices obtained from A and A' by deleting their first row have the same row space and are in reduced echelon form. By our inductive hypothesis, these two matrices are equal. It follows that $A_1 = A_1'$. $\square$

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