McGill University
MATH 251: Algebra 2
Assignment 8: due Wednesday, April 14, 2004

1. If P is a linear operator on a vector space V with $P^2 = P$ and $Q = 1 - P$, show that

$$V = \mathrm{Ker(P)} \oplus \mathrm{Ker(Q)} = \mathrm{Im(Q)} \oplus \mathrm{Im(P)}.$$

   If V is an inner product space, show that P is orthogonal projection on $W = \mathrm{Im(P)}$ if and only if P is self-adjoint.

2. (a) Show that $< A, B >= \mathrm{tr}(AB^t)$ defines an inner product on $\mathbb{R}^{m\times n}$.

   (b) Show that the linear operator T on $V = \mathbb{R}^{2\times 2}$ defined by

$$T(X) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X - X \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   is self-adjoint with respect to the inner product defined in part (a). Find an orthonormal basis of V consisting of eigenvectors of T.

3. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

   (a) Find an orthogonal matrix P such that P^tAP is a diagonal matrix.

   (b) Find the maximum and minimum values of the quadratic form $q(X) = X^tAX$ on the sphere $||X|| = 1$.

   (c) Find the spectral decomposition of A.

   (d) Find all symmetric matrices B with $B^2 = A$.

4. (a) Find the singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$.

   (b) Use this to find the generalized inverse of A.

5. Let A be the complex matrix $\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$.

   (a) Show that A is normal.

   (b) Find a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix.

   (c) Find the spectral decomposition of A.