McGill University
MATH 251: Algebra 2
Assignment 7: due Monday, April 5, 2004

1. Let A be the real matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

   (a) Find an invertible matrix P such that PAP^t is a diagonal matrix with diagonal entries ± 1 or 0.

   (b) What is the rank and signature of the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3^2 + 6x_1x_2 + 4x_1x_3 + 2x_2x_3?$$

   Find independent linear forms y_1, y_2, y_3 in x_1, x_2, x_3 such that

$$q(x) = a_1y_1^2 + a_2y_2^2 + a_3y_3^2$$

   with $a_i = \pm 1$ or 0.

   (c) If b is the symmetric bilinear form associated to q, find a basis $f = (f_1, f_2, f_3)$ of $\mathbb{R}^3$ such that $b(f_i, f_j) = 0$ for $i \neq j$ and $b(f_i, f_i) = \pm 1, 0$.

2. Show that the Hermitian quadratic form

$$q(x) = x_1\overline{x}_1 + 2x_2\overline{x}_2 + 7x_3\overline{x}_3 + ix_1\overline{x}_2 - ix_2\overline{x}_1 - ix_3\overline{x}_3 + ix_3\overline{x}_1 + (1-i)x_2\overline{x}_3 + (1+i)x_3\overline{x}_2.$$

   is positive definite and find linear forms y_1, y_2, y_3 in $x_1, x_2, x_3 \in \mathbb{C}$ such that

$$q(x) = |y_1|^2 + |y_2|^2 + |y_3|^2.$$

   If b is the associated Hermitian form, find a basis f_1, f_2, f_3 of $\mathbb{C}^3$ such that $b(f_i, f_j) = \delta_{ij}$.

3. Find the least-squares solution of smallest norm of the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + 5x_3 &= 2. \end{aligned}$$

4. Let $V = C([0,1])$ be the real inner product space of continuous real-valued functions on the interval $[0,1]$ where the inner product is given by

$$< f, g >= \int_0^1 f(x)g(x)\,dx.$$

   Find the best approximation to $h(x) = x^3$ by a function in $W = \mathrm{Span}(1, \mathrm{x}, \mathrm{x}^2)$.