McGill University
MATH 251: Algebra 2
Assignment 6: due Monday, March 22, 2004

1. (a) Let $A \in F^{n\times n}$ be the matrix all of whose entries are equal to 1. Prove that A is diagonalizable if and only if the characteristic of the field F does not divide n. Find the Jordan canonical form of A and an invertible matrix P such that $P^{-1}AP$ is in Jordan canonical form.

   (b) Let T be a linear operator on a vector space V. If rank(T) = n, prove that T has at most $n+1$ distinct eigenvalues.

2. Let V be a finite-dimensional vector space over a field F.

   (a) If T is a diagonalizable operator on V, show that the restriction of T to any T-invariant subspace W of V is also diagonalizable.

   (b) If S, T are commuting operators on V, show that each eigenspace $\mathrm{Ker(T-a)}$ of T is S-invariant.

   (c) If S, T are commuting diagonalizable operators on V, prove that the are simultaneously diagonalizable; in other words, find a basis e of V such that $[T]_e$ and $[S]_e$ are diagonal matrices.

3. Find the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{F}_2^{6\times 6}$$

   and an invertible matrix P such that $P^{-1}AP$ is in Jordan canonical form.

4. (a) Determine whether or not the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & -1 \\ -6 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

   are similar or not. If they are, find an invertible matrix P with $P^{-1}BP = A$

   (b) Same question as (a) but with $B = \begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$.

   (c) Find the Jordan canonical form of the $n \times n$ matrix $A = [a_{ij}]$ with

$$a_{ij} = 0 \text{ for } i > j, \quad a_{11} = a_{22} = \cdots = a_{nn} = c, \quad a_{12}, a_{23}, \ldots, a_{(n-1)n} \neq 0.$$