McGill University
MATH 251: Algebra 2
Assignment 5: due Monday, March 15, 2004

1. (a) Let U, V be vector spaces over a field F and let

$$U_1 = \{(u, 0) \in U \times V \mid u \in U\}, \quad V_1 = \{(0, v) \in U \times V \mid v \in V\}.$$

Show that U_1, V_1 are subspaces of $U \times V$ isomorphic to U, V respectively and that $U \times V = U_1 \oplus V_1$.

(b) If W is a vector space over the field F and U, V are subspaces of W, show that the mapping $T : U \times V \to U + V$ defined by

$$T(u, v) = u + v$$

is a surjective linear mapping with kernel $\{(w, -w) \mid w \in U \cap V\}$ which is isomorphic to $U \cap V$. Show how one deduces that

$$\dim(U + V) + \dim(U \cap V) = \dim(U) + \dim(V).$$

(c) In (b) show that $B_1 \cup B_2 \cup B_3$, where B_1 is a basis for $U \cap V$ and B_2, B_3 complete B_1 to bases for U, V respectively, is a basis for $U + V$.

2. If $A = [a_{ij}] \in F^{n \times n}$ the trace of A is the scalar $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

(a) Show that the trace is a linear form on $F^{n \times n}$ and that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in F^{n \times n}$.

(b) If ϕ is any linear form on $F^{n \times n}$ such that $\phi(AB) = \phi(BA)$ for all $A, B \in F^{n \times n}$, prove that ϕ is a scalar multiple of the trace tr. **Hint:** Show that the subspace of $F^{n \times n}$ spanned by the matrices of the form $AB - BA$ has codimension 1.

(c) If T is a linear operator on an n-dimensional vector space V and A, B are matrices of T with respect to two different bases, show that $\mathrm{tr}(A) = \mathrm{tr}(B)$. This common value $\mathrm{tr}(T)$ is called the trace of T. Show that tr is a linear form on $E(V) = \mathrm{Lin}(V, V)$.

3. (a) Find a basis for the annihilator of the subspace of $\mathbb{F}_2^5$ spanned by the vectors

$$(1, 1, 1, 1, 1), (1, 0, 1, 1, 0), (0, 1, 0, 1, 0).$$

(b) If U, V are subspaces of a vector space W, prove that $(U + V)^0 = U^0 \cap V^0$. If W is finite-dimensional, show that $(U \cap V)^0 = U^0 + V^0$.

4. Let $T : U \to V$ be a linear mapping vector spaces over a field F and let $T^t : V^* \to U^*$ be the transpose mapping.

(a) Prove that $(\mathrm{Im}(T))^0 = \mathrm{Ker}(T^t)$ and $(\mathrm{Im}(T^t))^0 = \mathrm{Ker}(T)$.

(b) Show how (a) can be used to prove $\mathrm{rank}(T) = \mathrm{rank}(T^t)$ when U or V is finite-dimensional.