McGill University
MATH 251: Algebra 2
Assignment 4: due Wednesday, February 18, 2004

1. Using the decomposition theorem, find all solutions $y = f(x)$ of the differential equation

$$\frac{d^4y}{dx^4} - 6\frac{d^3y}{dx^3} + 13\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0.$$

Find the solution $y = f(x)$ satisfying $f(0) = f'(0) = f''(0) = f'''(0) = 1$.

2. Let F be a field and let $a \in F$, $a \neq 0$. For $i \in \mathbb{N}$, define $f_i \in V = F^{\mathbb{N}}$ by $f_i(n) = n^i a^n$, and let $V_k = \mathrm{span}(f_0, f_1, \ldots, f_{k-1})$. Let L be the left-shift operator on V.

   (a) Show that $(L-a)(f_0) = 0$, $(L-a)(f_1) = af_0$ and, for $i \geq 2$,

   $$(L-a)f_i = iaf_{i-1} + g_i \text{ with } g_i \in V_{i-1}.$$

   Deduce that $(L-a)^i(f_i) = i!a^i f_0$.

   (b) Using (a), show that $V_k \subseteq \mathrm{Ker}((L-a)^k)$.

   (c) If the characteristic of F is zero, prove that $f_0, f_1, \ldots, f_{k-1}$ are linearly independent. In this case, deduce that $V_k = \mathrm{Ker}((L-a)^k)$ using the fact that $\dim \mathrm{Ker}((L-a)^k) = k$. Note that $f_1 = f_2$ if $F = \mathbb{F}_2$.

   (d) Define $g_i \in V$ by $g_i(n) = \binom{n}{i} a^{n-i}$, where

   $$\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!} \in \mathbb{Z}.$$

   Show that $(L-a)(g_0) = 0$ and $(L-a)(g_i) = g_{i-1}$ for $i > 0$ and use this to show that

   $$\mathrm{Ker}((L-a)^k) = \mathrm{span}(g_0, g_1, \ldots, g_{k-1}).$$

   Moreover, show that any $f \in \mathrm{Ker}((L-a)^k)$ can be uniquely written in the form

   $$f = c_0 g_0 + c_1 g_1 + \cdots + c_{k-1} g_{k-1},$$

   where $c_i = (L-a)^i(f)(0)$.

   (e) Find a formula for $s_n = \sum_{k=1}^{n} k^2 2^k$.

3. Let $V_1, \ldots, V_n$ be subspaces of a vector space V such that $V = V_1 + \cdots + V_n$. If W_i is the sum of the subspaces V_j with $j \neq i$, the sum $V = V_1 + \cdots + V_n$ is said to be direct if $V_i \cap W_i = \{0\}$ for $1 \leq i \leq n$. Show that the following statements are equivalent.

   (a) The sum $V = V_1 + \cdots + V_n$ is direct;

   (b) If $v_1 + v_2 + \cdots + v_n = 0$ with $v_i \in V_i$ then $v_1 = v_2 = \cdots = v_n = 0$;

   (c) Every $v \in V$ can be uniquely written as a sum $v_1 + v_2 + \cdots + v_n$ with $v_i \in V_i$;

   (d) If B_i is a basis for V_i then $B_i \cap B_j = \emptyset$ for $i \neq j$ and $B = \cup_{i=1}^{n} B_i$ is a basis for V.
   Hint: Show that $(a) \implies (b) \implies (c) \implies (d) \implies (a)$.

4. If T is the operator in problem 2(b) of Assignment 3, show that $T^3 = 4T$. What are the eigenvalues of T? Find a basis of $\mathbb{R}^{2\times 2}$ such that the matrix of T with respect to this basis is a diagonal matrix. How is this matrix related to the matrix found in 2(c) of Assignment 3?