McGill University
MATH 251: Algebra 2
Assignment 3: due Monday, February 9, 2004

1. Determine whether the following subsets of $\mathbb{R}^{\mathbb{R}}$ are linearly independent or not.

   (a) $S = \{e^x, e^{2x}, e^{3x}\}$;

   (b) $S = \{\sin(x+1), \sin(x+2), \sin(x+3)\}$;

   (c) $S = \{(x+1)^2, (x+2)^2, (x+3)^2\}$.

2. Let $V = \mathbb{R}^{2\times 2}$, let $A \in V$ and let $T : V \to V$ be defined by $T(X) = AX - XA$.

   (a) Show that T is a linear mapping;

   (b) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find bases for the kernel and image of T;

   (c) If A is as in (b), find the matrix of T with respect to the basis

   $$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   What is the rank and nullity of this matrix?

3. Let $s_n = \sum_{k=0}^{n} k2^k$ and let $s = (s_0, s_1, \ldots, s_n, \ldots) \in \mathbb{R}^{\mathbb{N}}$.

   (a) Show that $s \in \mathrm{Ker}(L-1)(L-2)^2$, where L is the left shift operator;

   (b) Show that $u = (1, 1, \ldots 1, \ldots)$, $v = (1, 2, \ldots, 2^n, \ldots)$, $w = (0, 2, \ldots, n2^n, \ldots)$ form a basis for $\mathrm{Ker}(L-1)(L-2)^2$; (**Hint:** Use problem 5 of assignment 1.)

   (c) Use (b) to find a formula for s_n.

4. Let $V = C^\infty(\mathbb{R})$ be the vector space of infinitely differentiable real valued functions on the real line. Let D be the differentiation operator on V and let E_a be the operator defined by $E_a(f(x)) = e^{ax}f(x)$.

   (a) Show that E_a is invertible with inverse E_{-a};

   (b) Show that $(D-a)E_a = E_aD$ and deduce that $(D-a)^n = E_aD^nE_a^{-1}$;

   (c) Using (b), find a basis for $\mathrm{Ker}(D-a)^n$.