McGill University
MATH 251: Algebra 2
Assignment 1: due Friday, January 16, 2004

1. Let V be a vector space over a field F, let V' be a set and let $f : V \to V'$ be a bijective mapping. Prove that there is a unique vector space structure on V' over F which makes f an isomorphism of vector spaces.

2. Let F be a field and let $V_{11} = F^{n\times n}$, $V_{12} = F^{n\times m}$, $V_{21} = F^{m\times n}$, $V_{22} = F^{m\times m}$. If $p = m + n$, prove that
$$\prod_{1\le i,j\le 2} V_{ij} \cong F^{p\times p}$$
as vector spaces over F by giving an explicit isomorphism.

3. If V is a vector space, prove that it cannot be the union of two proper subspaces. Can a vector space be the union of three proper subspaces?

4. Which of the following subsets S of the given vector space V are subspaces?
   (a) $V = \mathbb{R}^2$, $S = \{(x, y) \in V \mid x^2 + xy + y^2 = 0\}$;
   (b) $V = \mathbb{R}^2$, $S = \{(x, y) \in V \mid x^2 + 2xy + y^2 = 0\}$;
   (c) $V = \mathbb{R}^2$, $S = \{(x, y) \in V \mid x^2 + 2xy - 8y^2 = 0\}$;
   (d) $V = \mathbb{R}^{\mathbb{N}}$, $S = \{x = (x_0, x_1, \ldots, x_n, \ldots) \in V \mid x_{n^2} = nx_n \text{ for } n \ge 0\}$;
   (e) $V = \mathbb{R}^{\mathbb{R}}$, $S = \{f \in V \mid f(x + 2) - f(x + 1) = xf(x) \text{ for all } x \in \mathbb{R}\}$.

5. Let F be a field and let $a_1, a_2, \ldots, a_k \in F$. Show that
$$S = \{x \in F^{\mathbb{N}} \mid x_{n+k} = a_1x_{n+k-1} + a_2x_{n+k-2} + a_3x_{n+k-3} + \cdots + a_kx_n \text{ for } n \ge 0\}$$
is a subspace of $F^{\mathbb{N}}$ and that the mapping $T : S \to F^k$ defined by
$$T(x) = (x_0, x_1, \ldots, x_{k-1})$$
is an isomorphism of vector spaces.