MATH 247 Midterm Test Solutions

1. (a) S is not a subspace since $u = (1, 1, 1, \ldots, 1, \ldots) \in S$ but $2u = (2, 2, 2, \ldots, 2, \ldots) \notin S$.

   (b) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then the zero matrix $0 \in S$ since $0A = A0 = 0$. If $X, Y \in S$ and $a, b \in \mathbb{R}$ then $(aX + bY)A = aXA + BYA = aAX + bAY = A(aX + bY)$ which implies $aX + bY \in \S$. Hence S is a subspace.

2. (a) Since $ae^x + be^{2x} + ce^{3x} + de^{4x} = 0$ in $\mathrm{Ker}(D-1) + \mathrm{Ker}(D-2) + \mathrm{Ker}(D-3) + \mathrm{Ker}(D-4)$ and $e^{kx} \in \mathrm{Ker}(D-k)$ we see that $ae^x = be^{2x} = ce^{3x} = de^{4x} = 0$ since the sum is direct. Setting $x = 0$, we get $a = b = c = d = 0$. Hence the given sequence is linearly independent.

   (b) Since A, A^2, A^3, A^4, A^5 is a sequence of length 5 in $\mathbb{R}^{2\times 2}$, a four dimensional vector space, we see that the given sequence is linearly dependent.

3. (a) T is linear since $T(af+bg)(x) = (af+bg(x)+(af+bg)(-x) = af(x)+bf(-x)+af(-x)+bg(-x) = a(f(x)+f(-x))+b(g(x)+g(-x)) = aT(f)(x)+bT(g)(x) = (aT(f)+bT(g))(x)$ which implies $T(af + bg) = aT(f) + bT(g)$.

   (b) Since If $f \in V$ then $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and so $Tf(x) = f(x) + f(-x) = 2a_0 + 2a_2x^2$ which implies that $1, x^2$ is a basis for the image of T. We also see that $T(f) = 0 \iff a_0 = a_2 = 0$ since $1, x^2$ is a linearly independent sequence so that x, x^3 is a basis for $\mathrm{Ker}(T)$.

4. (a) We have $(A-I)^2 = 4(A-I)$ so that $(A-I)(A-5I) = 0$. Hence the possible eigenvalues of A are $1, 5$, the roots of $(x-1)(x-5) = 0$. We have $\mathrm{Null(A-I)} = \mathrm{Null}(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$

   $\mathrm{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right)$, a vector space of dimension 3.

   Since $\mathbb{R}^{4\times 1} = \mathrm{Null((A-I)(A-5I)} = \mathrm{Null(A-I)} \oplus \mathrm{(A-5I)}$, we have

   $\dim \mathrm{Null(A-5I)} = 1$. Since $A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ this implies that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for $\mathrm{Null(A-5I)}$.

   Hence $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for $\mathbb{R}^{4\times 1}$ consisting of eigenvectors of A.

   (b) If $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, the matrix formed by the basis of eigenvectors of A, we have

$P^{-1}AP = \mathrm{diag}(1, 1, 1, 5)$, the 4×4 diagonal matrix with diagonal entries $1, 1, 1, 5$ which are the eigenvalues of the corresponding columns of P.