MATH 247: Solutions for Assignment 6

1. (a) Since the row sums of A are 5 we have that 5 is an eigenvalue with eigenvector $[1,1,1,1]^t$. Also, 3 is an eigenvalue since the rank of $A-3I$ is 2. Now

$$\text{Null}(A-3I) = \text{Span}\left(\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right), \quad \text{Null}(A-5I) = \text{Span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$$

so that $W = \text{Null}(A-3I) + \text{Null}(A-5I)$ is 3-dimensional. The orthogonal complement $W^\perp$ is spanned by $[1,-1,-1,1]^t$ which is an eigenvector of A with eigenvalue 1. Hence

$$f_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}, f_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, f_3 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, f_4 = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

is an orthonormal basis of V consisting of eigenvectors of A and $A = 3P_1 + 3P_2 + 5P_3 + P_4$ is spectral resolution of A with

$$P_1 = f_1f_1^t = \begin{bmatrix}0&0&0&0\\0&1/2&-1/2&0\\0&-1/2&1/2&0\\0&0&0&0\end{bmatrix}, \quad P_2 = f_2f_2^t = \begin{bmatrix}1/2&0&0&-1/2\\0&0&0&0\\0&0&0&0\\-1/2&0&0&1/2\end{bmatrix},$$

$$P_3 = f_3f_3^t = \begin{bmatrix}1/4&1/4&1/4&1/4\\1/4&1/4&1/4&1/4\\1/4&1/4&1/4&1/4\\1/4&1/4&1/4&1/4\end{bmatrix}, \quad P_4 = f_4f_4^t = \begin{bmatrix}1/4&-1/4&-1/4&1/4\\-1/4&1/4&1/4&-1/4\\-1/4&1/4&1/4&-1/4\\1/4&-1/4&-1/4&1/4\end{bmatrix}.$$

The spectral decomposition of A is $A = 3Q_1 + 5Q_2 + Q_3$ where $Q_1 = P_1 + P_2$, $Q_2 = P_3$, $Q_3 = P_4$.

(b) The solution is $X = e^{-itA}X(0)$ with $X(0) = [1,1,1,1]^t$. By the spectral decomposition, we have $e^{-itA} = e^{-3it}Q_1 + e^{-5it}Q_2 + e^{-it}Q_3$ so that

$$X = (e^{-3it}Q_1 + e^{-5it}Q_2 + e^{-it}Q_3)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = e^{-5it}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

(c) We have $q(X) = X^AX = Y^tP^tAPY$ if $X = PY$. Choosing for P the orthogonal matrix whose columns are f_1, f_2, f_3, f_4 found in (a), we have $q(X) = 3y_1^2 + 3y_2^2 + 5y_3^2 + 5y_4^2$ with $||X|| = ||Y||$. If $||X|| = 1$ we have $1 \le q(X) \le 5$ with $q(X) = 1$ if and only if $X = \pm f_4$ and $q(X) = 5$ if and only if $X = \pm f_3$.

2. We have

$$A - I = \begin{bmatrix}0&-1&0&0&1\\0&0&1&-1&1\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}, \ (A-I)^2 = \begin{bmatrix}0&0&-1&1&-1\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}, \ (A-I)^3 = \begin{bmatrix}0&0&0&0&-1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},$$

$(A-I)^4 = 0$ so that $\text{Null}(A-I) = \text{Span}(e_1, e_3+e_4)$, $\text{Null}((A-I)^2) = \text{Span}(e_1, e_2, e_3+e_4)$, $\text{Null}((A-I)^3) = \text{Span}(e_1,e_2,e_3,e_4)$, $\text{Null}((A-I)^4) = \mathbb{R}^{5\times 1}$. Hence e_5 generates the cycle

$$f_1 = (A-I)^3e_5 = -e_1, \ f_2 = (A-I)^2e_5 = -e_1+e_2, \ f_3 = (A-I)e_5 = e_1+e_2+e_3, f_4 = e_5.$$

of length 4. Completing f_1 to a basis $f_1, f_5 = e_3 + e_4$ of $\text{Null}(A - I)$, we obtain a cycle of length 1 generated by f_5. If P is the matrix whose columns are f_1, f_2, f_3, f_4, f_5 we have

$$P = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the Jordan canonical form of A.

3. Since $A^2 = 4I$ and $A \neq \pm 2I$ the eigenvalues of A are ± 2. We have

$$\text{Null}(A - 2I) = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right), \ \text{Null}(A + 2I) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right).$$

Applying the Gram-Schmidt process to the basis for $\text{Null}(A + 2I)$ we get

$$f_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ f_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \ f_3 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

which, together with $f_4 = [-1/2, 1/2, 1/2, 1/2]^t$, yields an orthonormal basis of $\mathbb{R}^{4\times 1}$ consisting of eigenvectors of A. The spectral decomposition of A is $A = -2Q_1 + 2Q_2$ where $Q_1 = f_1 f_1^t + f_2 f_2^t + f_3 f_3^t$ and $Q_2 = f_3 f_3^t$.

4. (a) The statement is false since $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are symmetric while their product $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not.

   (b) If λ is an eigenvalue of T we have $\lambda^9 = \lambda^8$ which implies $\lambda = 0, 1$. Hence T is self-adjoint since the eigenvalues are real. Since $\lambda^2 = \lambda$ for any eigenvalue of T the spectral theorem implies that $T^2 = T$.

5. (a) If $T(v) = \lambda v$ with $\lambda \neq 0$ we have $v = T((1/\lambda)v)$ so that eigenvectors of T with non-zero eigenvalues are in the image of T. Since eigenvectors of T with distinct eigenvalues are linearly independent, the number of distinct non-zero eigenvalues of T is at most $n = \dim(V)$ and so the number of distinct eigenvalues of T is at most $n + 1$.

   (b) If u, v are linearly independent and $T(u) = au$, $T(v) = bv$, $T(u + v) = c(u + v)$ we have $cu + cv = au + bv$ so that $a = b = c$. Since any basis of V is a basis of eigenvectors for T this implies that the eigenvalues of the basis vectors are all equal. Hence $T(v) = cv$ for some scalar c independent of v.