MATH 247: Solutions for Assignment 5

1. (a) If $A = [a_{ij}], B = [b_{ij}]$ we have $T(A+B) = \sum_{i=1}^n (a_{ii}+b_{ii}) = \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} = \text{tr}(A) + \text{tr}(B)$ and $\text{tr}(cA) = \sum_{i=1}^n c\,a_{ii} = c\sum_{i=1}^n a_{ii} = c\,\text{tr}(A)$ for any scalar c. Also $\text{tr}(AB) = \sum_{i,j=1}^n a_{ij}b_{ji} = \text{tr}(BA)$.

   (b) If E_{ij} is the $n\times n$ matrix with (i,j)-th entry 1 and all other entries equal to 0, we have $E_{ij}E_{jk} = E_{ik}$ and $E_{ij}E_{km} = 0$ if $j \neq k$. The kernel of the trace map has dimension n^2-1 with basis the matrices E_{ij} $(i \neq j)$, $E_{11} - E_{ii}$ $(i \neq 1)$. If $i \neq j$ we have $E_{ij} = E_{ii}E_{ij} - E_{ij}E_{ii}$ and we have $E_{11} - E_{ii} = E_{1i}E_{i1} - E_{i1}E_{1i}$ which shows that Ker(tr) is spanned by matrices of the form $AB - BA$. This shows that $\text{tr}(A) = 0$ implies $\phi(A) = 0$. If A is any $n\times n$ matrix we have $A = \text{tr}(A)E_{11} + C$ with $\text{tr}(C) = 0$ which implies $\phi(A) = c\,\text{tr}(A)$ with $c = \phi(E_{11})$ and hence $\phi = c\,\text{tr}$.

2. (a) For fixed (y_1,y_2) we have that $\langle (x_1,x_2),(y_1,y_2)\rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$ is linear in (x_1,x_2) and $\langle (x_1,x_2),(y_1,y_2)\rangle = \langle (y_1,y_2),(x_1,x_2)\rangle$. Since $\langle (x_1,x_2),(x_1,x_2)\rangle = (x_1+2x_2)^2 + x_2^2$ we see that $\langle (x_1,x_2),(x_1,x_2)\rangle \geq 0$ with equality if and only if $(x_1,x_2) = (0,0)$.

   (b) For fixed (y_1,y_2) we have that $\langle (x_1,x_2),(y_1,y_2)\rangle = x_1\bar{y}_1 + ix_1\bar{y}_2 - ix_2\bar{y}_1 + 2x_2\bar{y}_2$ is linear in (x_1,x_2) and $\langle (x_1,x_2),(y_1,y_2)\rangle = \overline{\langle (y_1,y_2),(x_1,x_2)\rangle}$. Since $\langle (x_1,x_2),(x_1,x_2)\rangle = |x_1 - ix_2|^2 + |x_2|^2$ we see that $\langle (x_1,x_2),(x_1,x_2)\rangle \geq 0$ with equality if and only if $(x_1,x_2) = (0,0)$.

3. (a) An orthonormal basis for $W = \text{Span}(1,x,x^2)$ is $f_1(x) = 1$, $f_2(x) = x - \frac{1}{2}$, $f_3(x) = x^2 - x + \frac{1}{6}$. We have $||f_1|| = 1$, $||f_2|| = 1/2\sqrt{3}$, $||f_3|| = 1/6\sqrt{5}$. The best approximation to $h(x) = \sin(\pi x)$ by a function in W is the orthogonal projection of h on W. This is the function
$h(x) = \frac{\langle h,f_1\rangle}{\langle f_1,f_1\rangle}f_1 + \frac{\langle h,f_2\rangle}{\langle f_2,f_2\rangle}f_2 + \frac{\langle h,f_3\rangle}{\langle f_3,f_3\rangle}f_3 = \frac{2}{\pi} + \frac{60(\pi^2-12)}{\pi^3}(x^2 - x + \frac{1}{6})$.

   (b) If $g = af_1 + bf_2 + cf_3$ then $f(1) = \langle f,g\rangle$ for all $f \in W$ gives $1 = f_1(1) = a$, $1/2 = b/12$, $1/6 = c/180$ so that $a = 1$, $b = 6$, $c = 30$. Hence $g(x) = 30x^2 - 24x + 3$.

4. (a) First note that $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ is self-adjoint. Then $\langle X, T(Y)\rangle = \text{tr}(X(AY - YA)^*) = \text{tr}(X(AY^* - Y^*A)) = \text{tr}(XAY^*) - \text{tr}(XY^*A) = \text{tr}(XAY^*) - \text{tr}(AXY^*) = \text{tr}((AX - XA)Y^*) = \langle T(X), Y\rangle$.

   (b) If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $T(X) = i\begin{bmatrix} b+c & d-a \\ d-a & -b-c \end{bmatrix} = i(b+c)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i(d-a)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which implies $\text{Im}(T) = \text{Span}(E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ and $\text{Ker}(T) = \text{Span}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$. The subspace $W = \text{Im}(T)$ is T-invariant and $T(E) = -2iF$, $T(F) = 2iE$ implies that the matrix of the restriction of T to W with respect to the basis E, F is the matrix $B = \begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix}$. The minimal polynomial of B is x^2- so that $2, -2$ are the eigenvalues of B. Eigenvectors for these eigenvalues are $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ respectively. Thus the matrices $iE + F$ and $E + iF$ are eigenvectors of T with eigenvalues $2, -2$ respectively. It follows that the required orthonormal basis of eigenvectors of T is

$$F_1 = \frac{1}{2}\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix},\ F_2 = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix},\ F_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ F_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   (c) The matrix of T with respect to the basis $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, F_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is the matrix $A = \begin{bmatrix} 0 & i & i & 0 \\ -i & 0 & 0 & i \\ -i & 0 & 0 & i \\ 0 & -i & -i & 0 \end{bmatrix}$. The unitary matrix $U = \begin{bmatrix} i/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & i/2 & 0 & 1/\sqrt{2} \\ 1/2 & i/2 & 0 & -1/\sqrt{2} \\ -i/2 & -1/2 & 1/\sqrt{2} & 0 \end{bmatrix}$ whose columns are the coordinate vectors of F_1, F_2, F_3, F_4 satisfies $U^{-1}AU = \text{diag}(2,-2,0,0)$, the diagonal matrix with entries $2, -2, 0, 0$, since $T(F_1) = 2F_2$, $T(F_2) = -2F_2$, $T(F_3) = T(F_4) = 0$.