Math 247: Honours Applied Linear Algebra
Solutions to Assignment 4

1. (a) $A^2-(a+d)A+(ad-bc)I=\begin{bmatrix}a^2+bc-(a+d)a+ad-bc & ab+bd-(a+d)b\\ ac+cd-(a+d)c & bc+d^2-(a+d)d+ad-bc\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$.

   (b) If $m(x)=x^2-(a+d)x+(ad-bc)$ we have $m(A)=0$ by (a) and so the minimal polynomial $m_A(x)$ of A divides $m(x)$. Since A is not a scalar multiple of the identity matrix the degree of $m_A(x)$ is not 1 and so $m_A(x)=m(x)$ since both polynomials are monic and of the same degree.

   (c) If $m_A(x)=(x-\lambda)^2$ we have $A-\lambda I\neq 0$ which implies the existence of a column matrix P_2 with $P_1=(A-\lambda I)P_2\neq 0$.Then $AP_1=\lambda P_1$, $AP_2=P_1+\lambda P_1$. If P is the matrix whose columns are P_1,P_2 then P is invertible since P_2 is not a scalar multiple of P_1 and $P^{-1}AP=\begin{bmatrix}\lambda&1\\0&\lambda\end{bmatrix}=B$ since B is the matrix of the linear operator T_A on $\mathbb{F}^{2\times 1}$ defined by $T_A(X)=AX$ with respect to the basis P_1,P_2. Conversely if $P^{-1}AP=B$ then B is the matrix of T_A with respect to the basis formed by the columns of P so that the minimal polynomial of T_A is equal to $m_B(x)=(x-\lambda)^2$.

2. We are given that $x_{n+1}=x_n/2+y_n/3$, $y_{n+1}=x_n/2+2y_n/3$ which proves the assertion. It follows by induction that

$$\begin{bmatrix}x_n\\y_n\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{2}{3}\end{bmatrix}^{n-1}\begin{bmatrix}1/2\\1/2\end{bmatrix}.$$

The minimal polynomial of $A=\begin{bmatrix}\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{2}{3}\end{bmatrix}$ is $x^2-(7/6)x+1/6=(x-1)(x-1/6)$ so that the eigenvalues of A are 1 and 1/6. Since $\mathrm{Null}(A-1)=\mathrm{Span}(\begin{bmatrix}2\\3\end{bmatrix}$ and $\mathrm{Null}(A-(1/6)I)=\mathrm{Span}(\begin{bmatrix}1\\-1\end{bmatrix}$ we see that $P^{-1}AP=\begin{bmatrix}1&0\\0&1/6\end{bmatrix}$ where $P=\begin{bmatrix}2&1\\3&-1\end{bmatrix}$. Hence $A=P\begin{bmatrix}1&0\\0&1/6\end{bmatrix}P^{-1}$ so that $A^n=P\begin{bmatrix}1&0\\0&1/6\end{bmatrix}^nP^{-1}=P\begin{bmatrix}1&0\\0&1/6^n\end{bmatrix}P^{-1}$. Hence $\lim_{n\to\infty}A^n=P\begin{bmatrix}1&0\\0&1/6\end{bmatrix}^nP^{-1}=P\begin{bmatrix}1&0\\0&0\end{bmatrix}P^{-1}=\begin{bmatrix}\frac{2}{5}&\frac{2}{5}\\\frac{3}{5}&\frac{3}{5}\end{bmatrix}$ so that $\lim_{n\to\infty}\begin{bmatrix}x_n\\y_n\end{bmatrix}=\begin{bmatrix}\frac{2}{5}&\frac{2}{5}\\\frac{3}{5}&\frac{3}{5}\end{bmatrix}\begin{bmatrix}1/2\\1/2\end{bmatrix}=\begin{bmatrix}2/5\\3/5\end{bmatrix}$.

3. Since $m_A(x)=x^2-(a+d)x+ad-bc$ we have $m_A(1)=1-(a+d)+ad-bc=1-(a+1-b)+a(1-b)-b(1-a)=0$. Hence $m_A(x)=(x-1)(x-r)$. Since $a+d=1+r$ and $0<a+d<2$ we have $-1<r<1$. Since $m_A(x)$ is a product of distinct linear factors there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix with diagonal entries $1,r$. It follows as above that $B=\lim_{n\to\infty}A^n$ exists. Since $AB=B$ the columns of B are eigenvectors of A with eigenvalue 1. Also, since the product of stochastic matrices is stochastic, it follows that B is stochastic. Hence, if $\begin{bmatrix}\alpha\\\beta\end{bmatrix}$ is the first column of B it is an eigenvector of A with $X=\alpha+\beta=1$. The second column of B must be cX for some scalar c since $\dim\mathrm{Null}(A-I)=1$. Now $c\alpha+c\beta=1$ implies $c=1$.

4. (a) We have $(1+T+T^2+\cdots+T^{n-1})(1-T)=(1-T)(1+T+T^2+\cdots+T^{n-1})=$
   $1+T+T^2+\cdots+T^{n-1}-(T+T^2+\cdots+T^n)=T^n=0$.

   (b) Since $(T-b)=(T-a+a-b)=(a-b)(1-S)$ with $S=(T-a)/(b-a)$. Since $S^n=0$ we see by (a) that $1-S$ is invertible and hence $T-b$ is invertible with $(T-b)^{-1}=(a-b)^{-1}(1+S+\ldots+S^{n-1})$.

   (c) Since $W=\mathrm{Null}(D-2)^2=\mathrm{Span}(e^{2x}xe^{2x})$ is D-invariant we can restrict D to W to get a linear operator R on W with $(R-2)^2=0$. By (b) the operator $R-1$ is invertible with $(R-1)^{-1}=1-(R-2)$. Hence $(R-1)^2$ is invertible with $(R-1)^2=(1-(R-2))^2=1-2(R-2)=5-2R$. Hence $y_P=(3-2R)(xe^2)=5xe^{2x}-2D(xe^{2x})=xe^{2x}-2e^{2x}$ is a solution of the given differential equation.

   (d) Since $(D-2)^2(y-y_P)=(D-2)^2y-(D-2)^2y_P=0$ we have $y-y_P\in\mathrm{Span}(e^{2x},xe^{2x})$ so that $y=y_P+Ae^{2x}+Bxe^{2x}$.

5. We have

$$W_1 = \mathrm{Null}(A-I) = \mathrm{Span}\left(X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right),\ W_2 = \mathrm{Null}(A+I) = \mathrm{Span}\left(X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, X_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right)$$

and $BX_1 = X_1 + 4X_2, BX_2 = 2X_2 + 3X_2, BX_3 = 2X_3 + X_4, BX_4 = X_3 + 2X_4$. If $T_(X) = BX$ and R_1, R_2 are the restrictions of T to W_1 and W_2 we have

$$C = [R_1]_{(X_1,X_2)} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad D = [R_2]_{(X_3,X_4)} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Now $m_C(x) = x^2 - 4x - 5 = (x-5)(x+1)$, $m_D(x) = x^2 - 2x - 3 = (x-3)(x+1)$ so that C and D are diagonalizable. Since

$$\mathrm{Null}(C-5I) = \mathrm{Span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix}),\ \mathrm{Null}(C+I) = \mathrm{Span}(\begin{bmatrix} 1 \\ -1 \end{bmatrix}),\ \mathrm{Null}(D-3I) = \mathrm{Span}(\begin{bmatrix} 1 \\ 1 \end{bmatrix}),\ \mathrm{Null}(C+I) = \mathrm{Span}(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$$

we see that

$$P_1 = x_1 + 2x_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -2 \end{bmatrix},\ P_2 = X_1 - X_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix},\ P_3 = X_3 + X_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix},\ P_4 = X_3 - X_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

are eigenvectors of B with eigenvalues $5, -1, 3, -1$ respectively. Since they are also eigenvectors of A with eigenvalues $1, 1, -1, -1$ respectively, we see that the matrix P whose columns are P_1, P_3, P_3, P_4 is an invertible matrix which diagonalizes both A and B.

6. (a) If $u \in \mathrm{Ker}(p(S))$ then $p(S)(u) = 0$ which implies $p(S)T(u) = Tp(S)(u) = 0$ so that $T(u) \in \mathrm{Ker}(p(S))$. If $u \in \mathrm{Im}(p(S))$ then $u = p(S)(v)$ for some v and $T(u) = Tp(S)(v) = p(S)T(v) \in \mathrm{Im}(p(S)$.
   (b) Let R be the restriction of T to W. Since W is T-invariant, R is a linear operator on W. Since $m_T(R) = 0$ we have $m_R(x) \mid m_T(x)$. But T diagonalizable implies that $m_T(x)$ is a product of distinct linear factors and hence the same is true of $m_R(x)$. Thus R is diagonalizable.
   (c) Since S is diagonalizable V is the direct sum of the eigenspaces of S. Since T commutes with S each of the eigenspaces of S are T-invariant. Hence, using (b), we see that T is diagonalizable if and only if the restriction of T to each eigenspace of S is diagonalizable.

7. (a) Let $W_k = \mathrm{Ker}((L-a)^k) = \mathrm{Span}((a^{n-1}), (na^{n-1}), \ldots, (n^{k-1}a^{n-1}))$. Then $(L-a)(W_k) \subseteq W_{k-1}$ since

$$(L-1)((n^{i-1}a^{n-1})) = (((n+1)^{i-1}a^n - n^{i-1}a^n) = (\sum_{j<i} c_j n^j a_n) \in W_{i-1}$$

   for $i \geq 1$. It follows that $(L-a)^k(W_k) \subseteq W_0 = \{0\}$. Hence $W_k \subseteq \mathrm{Ker}(L-a)^k$ and we must have equality since $\dim \mathrm{Ker}(L-a)^k = k = \dim W_k$ since $(a^{n-1}), (na^{n-1}), \ldots, (n^{k-1}a^{n-1})$ is a linearly independent sequence. Indeed, $A_1 a^{n-1} + A_2 n a^{n-1} + \ldots + A_k n^{k-1} a^{n-1} = 0$ for all $n \geq 1$ if and only if $A_1 + A_2 n + \ldots + A_k n^{k-1} = 0$ for all $n \geq 1$ which implies $A_1 = \cdots = A_k = 0$ since otherwise $A_1 + A_2 x + \cdots + A_k x^{k-1}$ would be a polynomial of degree $\leq k-1$ with more than $k-1$ roots.
   (b) $(L-1)(s) = ((n+1)^2 3^{n+1}) = 9(n^2 3^{n-1} + 18(n3^{n-1}) + 9(3^{n-1}) \in \mathrm{Ker}(L-3)^3 \implies$ $s \in \mathrm{Ker}(L-1)^2(L-1)) = \mathrm{Span}((1), (3^{n-1}), (n3^{n-1}), (n^2 3^{n-1})) \implies s_n = A + B3^{n-1} + Cn3^{n-1} + Dn^2 3^{n-1}$ for unique scalars A, B, C, D. Using the fact that $s_1 = 3, s_2 = 39, s_3 = 282, s_4 = 1578$ we get $A = -3/2, B = 9/2, C = -9/2, D = 9/2$.

8. We have $\frac{dX}{dt} = AX$ with $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ and $m_A(x) = x^2 + 4x + 3 = (x+1)(x+3)$, $\mathrm{Null}(A+I) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathrm{Null}(A+3I) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Setting $X = PY$ with $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ we get $\frac{dY}{dt} = P^{-1}APY$ from which $\frac{dy_1}{dt} = -y_1$, $\frac{dy_2}{dt} = -3y_2$ and so $y_1 = Ae^{-x}$, $y_2 = Be^{-3x}$. Hence $x_1 = y_1 + y_2 = Ae^{-x} + Be^{-3x}$, $x_2 = y_1 - y_2 = Ae^{-x} - Be^{-2x}$.