McGill University
Solution Sheet for MATH 247 Assignment 3

1. (a) Since $(D-a)(x^k e^{ax}) = D(x^k e^{ax}) - ax^k e^{ax} = kx^{k-1}e^{ax} + ax^k e^{ax} - ax^k e^{ax} = kx^{k-1}e^{ax}$ we have, by induction, $(D-a)^k(x^k e^{ax}) = k!e^{ax}$ which implies $(D-a)^{k+1}(e^{ax}) = 0$ and hence that $(D-a)^n(x^k e^{ax}) = 0$ for $n > k$. Hence $U = \mathrm{Span}(\mathrm{e^{ax}}, \ldots, \mathrm{x^{k-1}e^{ax}}) \subseteq \mathrm{Ker((D-a)^n)}$. Since $\dim(\mathrm{Ker}((D-a)^n)) = n$ we have equality if $\dim(U) = n$. To prove the latter we have to show that $(e^{ax}, \ldots, x^{k-1}e^{ax})$ is linearly independent. If $a_0, \ldots, a_{n-1}$ are scalars with $a_0 e^{ax} + a_1 x e^{ax} + \cdots + a_{n-1}x^{n-1}e^{ax} = 0$ for all x we have $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} = 0$ for all x. Differentiating n times and setting $x = 0$ in the resulting equations, we get $a_i = 0$ for all i.

   Second Solution. If P is the operator on V defined by $P(f)(x) = e^{ax}f(x)$ we have $(D-a)P = PD$ so that $D-a = P^{-1}DP$. Hence $(D-a)^n = PD^nP^{-1}$. It follows that $f \in \mathrm{Ker}((D-a)^n)$ if and only if $P^{-1}(f) \in \mathrm{Ker}(D^n)$. Using the fact that $\mathrm{Ker}(D^n) = \mathrm{Span}(1, \mathrm{x}, \ldots, \mathrm{x^{n-1}})$, we get $f \in \mathrm{Ker}((D-a)^n)$ if and only if $e^{-ax}f(x) \in \mathrm{Span}(1, \mathrm{x}, \ldots, \mathrm{x^{n-1}})$.

   (b) We have $f^{(iv)}(x) - 2f''(x) + f(x) = 0$ if and only if $f \in \mathrm{Ker}(D^4 - 2D^2 + 1) = \mathrm{Ker}((D-1)^2(D+1)^2 \supseteq \mathrm{Ker}((D-a)^2) + \mathrm{Ker}((D+1)^2)$. Since $\dim(W) = 4$ we have to show $\dim(\mathrm{Ker}((D-a)^2) + \mathrm{Ker}((D+1)^2)) = 4$ in order to show equality. Since $\mathrm{Ker}((D-a)^2) + \mathrm{Ker}((D+1)^2) = \mathrm{Span(e^x, xe^x)} + \mathrm{Span(e^{-x}, xe^{-x})} = \mathrm{Span(e^x, xe^x, e^{-x}, xe^{-x})}$ we have to show that $(e^x, xe^x, e^{-x}, xe^{-x})$ is linearly independent. But $ae^x + bxe^x + ce^{-x} + dxe^{-x} = 0$ for all x yields on differentiation 3 times the identities $(a+b)e^x + bxe^x + (-c+d)e^{-x} - dxe^{-x} = 0$, $(a+2b)e^x + bxe^x + (c-2d)e^{-x} + dxe^{-x} = 0$, $(a+3b)e^x + bxe^x + (-c+3d)e^{-x} - dxe^{-x} = 0$ which gives on setting $x = 0$ the equations $a + c = 0, a + b - c + d = 0, a + 2b + c - 2d = 0, a + 3b - c + 3d = 0$ which implies $a = b = c = d = 0$ by Gaussian elimination. This also shows that $\mathrm{Ker}((D-1)^2) \cap \mathrm{Ker}((D+1)^2) = \{0\}$ and hence that $W = \mathrm{Ker}((D-1)^2) \oplus \mathrm{Ker}((D+1)^2)$.

   (c) From (b) we have $f(x) = ae^x + cxe^x + ce^{-x} + dxe^{-x}$. Hence $f(0) = a + c = 1$, $f'(0) = a + b - c + d = 2$, $f''(0) = a + 2b + c - 2d = 3$, $f'''(0) = a + 3b - c + 3d = 4$ from which $a = b = 1$, $c = d = 0$ and hence $f(x) = e^x + xe^x$.

2. (a) We have $A \in \mathrm{Ker}(T) \iff 2A + A^t = 0 \iff A^t = -2A \iff A = 0$ so that $\mathrm{Ker}(T) = \{0\}$ with basis the empty list. Since T is $1-1$ is onto as $\dim(V) = 4$ with basis $\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)$ which is therefore also a basis for $\mathrm{Im}(T)$.

   (b) We have $T^2(A) = T(T(A)) = 4A + 2T(A)^t = 4A + 2(2A + A^t)^t = 5A + 4A^t$ so that $T^2(A) - 4T(A) + 3A = 5A + 4A^t - 8A - 4A^t + 3A = 0$ which gives $T^2 - 4T + 3 = 0$. Hence, if λ is an eigenvalue of T, we must have $\lambda^2 - 4\lambda + 3 = 0$ from which $\lambda = 1, 3$. Since $T(A) = A \iff A^t = -A$ we see that $A = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ is an eigenvector of T with eigenvalue 1. Since $T(A) = 3A \iff A^t = A$ we see that $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ is an eigenvector of T with eigenvalue 3. Hence $1, 3$ are the eigenvalues of T.

   (c) We have $V = \mathrm{Ker}(T^2 - 4T + 3) = \mathrm{Ker}((T-1)(T-3)) = \mathrm{Ker}(T-1) \oplus \mathrm{Ker}(T-3)$ and $\mathrm{Ker}(T-1) = \mathrm{Span}(\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix})$, $\mathrm{Ker}(T-3) = \mathrm{Span}(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix})$ which yields the basis $\left(\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}), \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)$ of V consisting of eigenvectors of T.

3. (a) Let $F_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, F_2 = \begin{bmatrix}1\\2\\1\\2\end{bmatrix}, F_3 = \begin{bmatrix}1\\1\\2\\2\end{bmatrix}, F_4 = \begin{bmatrix}1\\2\\3\\3\end{bmatrix}$ and let E_1, E_2, E_3, E_4 be the standard basis of $\mathbb{R}^{4\times 1}$. Then $F_1 = E_1 + E_2 + E_3 + E_4$, $F_2 = E_1 + 2E_2 + E_3 + 2E_4$, $F_3 = E_1 + E_2 + 2E_3 + 2E_4$, $F_4 = E_1 + 2E_2 + 3E_3 + 3E_4$. Solving for E_1, E_2, E_3, E_4, we get $E_1 = F_1 + F_3 - F_4$, $E_2 = F_1 - 2F_3 + F_4$, $E_3 = F_1 - F_2 - F_3 + F_4$, $E_4 = -2F_1 + F_2 + 2F_3 - F_4$. Hence the columns of A are $AE_1 = F_1 - 4F_3 - 2F_4$, $AE_2 = F_1 - 4F_3 + 2F_4$, $AE_3 = F_1 - F_2 - 2F_3 + 2F_4$, $AE_4 = -2F_1 + F_2 + 4F_3 - 2F_4$ which gives $A = \begin{bmatrix}1 & -1 & 0 & 1\\-1 & 1 & 1 & 0\\-1 & -1 & 2 & 1\\-1 & -1 & 1 & 2\end{bmatrix}$. If B is a matrix with $AF_i = BF_i$ for all i we would have $AE_i = BE_i$ for all i and hence $A = B$.

   (b) Since $A^2 - 3A + 2I = (A - I)(A - 2I) = 0$ we see that the eigenvalues λ of A are roots of $\lambda^2 - 3\lambda + 2 = 0$ and hence must be 1 or 2. Since F_1, F_3 are eigenvectors of A with eigenvalues $1, 2$ respectively we see that the eigenvalues of A are $1, 2$.

4. (a) Let $F_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $F_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ and complete F_1, F_2 to a basis F_1, F_2, F_3, F_4 of $\mathbb{R}^{4\times 1}$ where $F_3 = E_3, F_4 = E_4$. Then $E_1 = \frac{3}{5}F_1 + \frac{2}{5}F_2 + F_3 + 2F_4$, $E_2 = \frac{4}{5}F_1 - \frac{1}{5}F_2 - 2F_3 - 3F_4$, $E_3 = F_3$, $E_4 = F_4$. Let T be the linear mapping of $\mathbb{R}^{4\times 1}$defined by $T(y_1F_1 + y_2F_2 + y_3F_3 + y_4F_4) = y_3F_1 + y_4F_2$. Then $\mathrm{Ker}(T) = \mathrm{Im}(T) = \mathrm{Span(F_1, F_2)}$. The required matrix A is the matrix of T with respect to the usual basis of $\mathbb{R}^{4\times 1}$. The columns of A are $AE_1 = F_1 + 2F_2$, $AE_2 = -2F_1 - 3F_2$, $AE_3 = F_1$, $AE_4 = F_2$ so that $A = \begin{bmatrix} 9 & -14 & 1 & 4 \\ 8 & -13 & 2 & 3 \\ 7 & -12 & 3 & 2 \\ 6 & -11 & 4 & 1 \end{bmatrix}$. Note that the matrix A is not unique.

Second Solution: If $[x_1, x_2, x_3, x_4]$ is a row of A we must have $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$, $4x_1 + 3x_2 + 2x_3 + x_4 = 0$ which, by Gaussian elimination, has the general solution $x_1 = a + 2b, x_2 = -2a - 3b, x_3 = a, x_4 = b$. Hence A has the form $A = \begin{bmatrix} a_1 + 2b_1 & -2a_1 - 3b_1 & a_1 & b_1 \\ a_2 + 2b_2 & -2a_2 - 3b_2 & a_2 & b_2 \\ a_3 + 2b_3 & -2a_3 - 3b_3 & a_3 & b_3 \\ a_4 + 2b_4 & -2a_4 - 3b_4 & a_4 & b_4 \end{bmatrix}$. Choosing $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, we obtain the required matrix.

(b) Since $A^2 = 0$ and the null space of A is $\mathrm{Span(F_1, F_2)}$ we see that 0 is the only eigenvalue of A. Since the only eigenvectors of A lie in the null space of A, a two dimensional subspace of $\mathbb{R}^{4\times 1}$, we see that $\mathbb{R}^{4\times 1}$ does not have a basis consisting of eigenvectors of A.

5. (a) Since $A^2 = 1$ the possible eigenvalues of A are $1, -1$. Now $\mathrm{Null(A - I)} = \mathrm{Span}(\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix})$ and $\mathrm{Null(A + I)} = \mathrm{Span}(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix})$ so that $1, -1$ are the eigenvalues of A and a basis for each eigenspace is given above.

(b) The matrix $P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$ satisfies $AP = PD$ with $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. The matrix P is invertible since its columns are linearly independent. This follows from the fact that $\mathbb{R}^{4\times 1} = \mathrm{Null(A^2 - 1)} = \mathrm{Null((A - I)(A + I))} = \mathrm{Null(A - I)} \oplus \mathrm{Null(A + I)}$. Hence $P^{-1}AP = D$.