McGill University
Solution Sheet for MATH 247 Assignment 2

1. (a) Let $w_1 = v_1 - v_2$, $w_2 = v_2 - v_3$, $w_3 = v_3 - v_4$, $w_4 = v_4$. Then $\mathrm{Span}(\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3, \mathrm{w}_4) \subseteq \mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4)$. But $v_4 = w_4$, $v_3 = w_3 + w_4$, $v_2 = w_2 + w_3 + w_4$, $v_1 = w_1 + w_2 + w_3 + w_4$ implies that $\mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_2, \mathrm{v}_4) \subseteq \mathrm{Span}(\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3, \mathrm{w}_4)$ and hence that $\mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_2, \mathrm{v}_4) = \mathrm{Span}(\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3, \mathrm{w}_4)$.

   (b) We have $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4$. Hence $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0$ implies $a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0$ and hence that $a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0$ since $(v_1, 2_2, v_3, v_4)$ is linearly independent. But this immediately gives $a_1 = a_2 = a_3 = a_4 = 0$.

   (c) Setting $w_1 = v_1 - v_2$, $w_2 = v_2 - v_3$, $w_3 = v_3 - v_4$, $w_4 = v_4 - w_1$, we have $w_1 + w_2 + w_3 + w_4 = 0$.

   (d) Since the vectors $(v_1 + w, v_2 + w, v_3 + w, v_4 + w)$ are linearly dependent, there are scalars a_1, a_2, a_3, a_4 not all zero such that $a_1(v_1 + w) + a_2(v_2 + w) + a_3(v_3 + w) + a_4(v_4 + w) = 0$. But then $a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + (a_1 + a_2 + a_3 + a_4)w = 0$. We must have $a = a_1 + a_2 + a_3 + a_4 \neq 0$; otherwise, $a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0$ contradicting the independence of (v_1, v_2, v_3, v_4). But then $w = b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4$ with $b_i = -a_i a^{-1}$ which implies $w \in \mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4)$.

2. (a) Let $u_1 = (1, 1, 1, 1), u_2 = (1, 1, 2, 2), u_3 = (2, 2, 1, 1), u_4 = (1, 1, 3, 3), u_5 = (4, 4, 3, 3)$. Then $(u_1, u_2, u_3, u_4, u_5)$ is linearly dependent since any sequence of vectors in $\mathbb{R}^4$ of length 5 is linearly dependent. To find a non-trivial dependence relation we have $a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4 + a_5 u_5 = (0, 0, 0, 0)$ if and only if

$$\begin{array}{rl} a_1 + a_2 + 2a_3 + a_4 + a_5 & = 0 \\ a_1 + 2a_2 + a_3 + 3a_4 + 3a_5 & = 0 \end{array} \iff \begin{array}{rl} a_1 + a_2 + 2a_3 + a_4 + a_5 & = 0 \\ a_2 - a_3 + 2a_4 + 2a_5 & = 0 \end{array} \iff \begin{array}{rl} a_1 + 3a_3 - a_4 - a_5 & = 0 \\ a_2 - a_3 + 2a_4 + 2a_5 & = 0 \end{array}.$$

   Setting $a_3 = -1, a_4 = a_5 = 0$ we get $3u_1 - u_2 - a_3 = (0, 0, 0, 0)$ which could also have been seen by inspection.

   (b) Since $f, g, h \in \mathrm{Span}(\sin, \cos)$ we obtain that (f, g, h) is linearly dependent. To find a non-trivial dependence relation we use the fact that

$$a\sin(x+1) + b\sin(x+2) + c\sin(x+3) = (a\cos(1) + b\cos(2) + c\cos(3))\sin(x) + (a\sin(1) + b\sin(2) + c\sin(3))\cos(x).$$

   Hence it suffices to find a non-trivial solution of $a\cos(1) + b\cos(2) + c\cos(3) = a\sin(1) + b\sin(2) + c\sin(3) = 0$; e.g.,

$$a = \frac{\cos(3)\sin(2) - \cos(2)\sin(3)}{\cos(1)\sin(2) - \cos(3)\sin(1)}, \quad b = \frac{\cos(1)\sin(3) - \cos(3)\sin(1)}{\cos(1)\sin(2) - \cos(3)\sin(1)}, \quad c = 1.$$

   (c) Suppose $ae^x + bxe^{2x} + cx^2e^{3x} = 0$ for all x. Setting $x = 0$ we get $a = 0$ so that $bxe^{2x} + cx^2e^{3x} = 0$ for all x. Differentiating, we get $be^{2x} + 2bxe^{2x} + 2cxe^{3x} + 3cx^2e^{3x} = 0$ for all x. Setting $x = 0$, we get $b = 0$. Hence $cx^2e^{3x} = 0$. Setting $x = 1$, we get $ce^3 = 0$ from which $c = 0$. Hence (f, g, h) is linearly independent.

   (d) The relation $a(1, 2, 3, \ldots, n, \ldots) + b(1, 2^2, 3^2, \ldots, n^2, \ldots) + (1, 2^3, 3^3, \ldots, n^3, \ldots) = (0, 0, 0, \ldots, 0, \ldots)$ implies that $a + b + c = 0, 2a + 4b + 8c = 0, 3a + 9b + 27c = 0$ which implies $a = b = c = 0$ by Gaussian elimination. Hence the given sequence of vectors is linearly independent.

3. (a) The solution to 2(a) shows that $u_3, u_4, u_5 \in \mathrm{Span}(\mathrm{u}_1, \mathrm{u}_2)$. Since (u_1, u_2) is linearly dependent it is a basis for the subspace W spanned by u_1, u_2, u_3, u_4. Since $a(1, 0, 0, 0) + b(0, 0, 1, 0) = (a, 0, b, 0)$ is in W if and only if $a = b = 0$, we see that $u_1, u_2, (1, 0, 0, 0), (0, 0, 1, 0)$ is a linearly independent sequence and hence a basis of $\mathbb{R}^4$.

   (b) We have $\mathrm{Span}((3, 3, 7, 7), (7, 7, 3, 3)) \subseteq \mathrm{Span}(1, 1, 0, 0), (0, 0, 1, 1)) = \mathrm{W}$ which implies equality since all the subspaces are 2-dimensional. Hence $((3, 3, 7, 7), (7, 7, 3, 3))$ is a basis of W.

4. We have $a(2, 1, 1, 3) + b(1, 2, 2, 1) + c(3, 2, 1, 5)) + d(1, 4, 3, 5) + e(1, 1, 2, 2) + f(4, 3, 1, 7) = 0$ if and only if

$$\begin{array}{rl} 2a + b + 3c + d + e + 4f & = 0 \\ a + 2b + 2c + 4d + e + 3f & = 0 \\ a + 2b + c + 3d + 2e + f & = 0 \\ 3a + b + 5c + 5d + 2e + 7f & = 0 \end{array} \iff \begin{array}{rl} 2a + b + 3c + d + e + 4f & = 0 \\ 3b - c + 5d + 3e - 2f & = 0 \\ c + d - e + 2f & = 0 \\ 3d + e & = 0 \end{array}$$

using Gaussian elimination. Since there are solutions with $e = 1, f = 0$ and $f + 1, e = 0$, we see that $((2, 1, 1, 3), (1, 2, 2, 1), (3, 2, 1, 5), (1, 4, 3, 5))$ spans $U + V$. It is also a basis for $U + V$ since $e = f = 0$ implies $a = b = c = d = 0$. We also obtain that $a(2, 1, 1, 3) + b(1, 2, 2, 1) + c(3, 2, 1, 5)) = d(1, 4, 3, 5) + e(1, 1, 2, 2) + f(4, 3, 1, 7)$ if and only if $e = -3d$ which shows that $U \cap V$ consists of those linear combinations of the form $d(1, 4, 3, 5) - 3d(1, 1, 2, 2) + f(4, 3, 1, 7) = d(-2, 1, -3, -1) + f(4, 3, 1, 7)$ with d, f arbitrary. Hence $((-2, 1, -3, -1), (4, 3, 1, 7))$ spans $U \cap V$ and is a basis since $((-2, 1, -3, -1), (4, 3, 1, 7))$ is linearly independent.

5. If $x_n = r^{n-1}$ then $x_{n+4} = 5x_{n+2} - 4x_n$ for all $n \iff r^{n+3} = 5r^{n+1} - 4r^{n-1}$ for all $n \iff r^4 - 5r^2 + 4 = 0 \iff r = \pm 1, \pm 2$. This shows that x, y, z, w are in U. To show that they span U we only have to show that (x, y, z, w) is linearly independent since $\dim(U) = 4$. But

$$ax + by + cz + dw = 0 \implies \begin{aligned} a + b + c + d &= 0 \\ a - b + 2c - 2d &= 0 \\ a + b + 4c + 4d &= 0 \\ a - b + 8c - 8d &= 0 \end{aligned} \implies a = b = c = d = 0$$

by Gaussian elimination. If $u = (1, 2, 3, 4, \ldots) \in U$ we have $u = ax + by + cz + dw$ which implies

$$\begin{aligned} a + b + c + d &= 1 \\ a - b + 2c + 2d &= 2 \\ a + b + 4c + 4d &= 3 \\ a - b + 8c - 8d &= 4 \end{aligned}$$

which has the unique solution $a = 5/6$, $b = -1/2$, $c = 1/2$, $d = 1/6$. Hence

$$\begin{aligned} u_n &= \frac{5}{6} - \frac{1}{2}(-1)^{n-1} + \frac{1}{2}2^{n-1} + \frac{1}{6}(-2)^{n-1} \\ &= \frac{5}{6} + \frac{1}{2}(-1)^n + 2^{n-2} - \frac{1}{3}(-2)^{n-2}. \end{aligned}$$